Vol. 42, Issue-05, 2022
pp.230-241

A NON-LOCAL PROBLEM FOR A THIRD-ORDER EQUATION WITH MULTIPLE CHARACTERISTICS

Turgenev Azizjon Mamasalievich
Kokand State Pedagogical Institute

Asqaralieva Mukhtasarkhan Azizjon kizi
Kokand State Pedagogical Institute

Abstract. In the paper nonlocal problem for equation of the third order with multiple characteristics are consider

Keywords: boundary conditions, integral equations

The third-order equations with multiple characteristics are considered

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial t} = 0,$$

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 u}{\partial t^2} = 0.$$

The equation was first analyzed by the Italian mathematician E. Del Vecchio. Dalneyshie issledovanie etikh uravneniy svyazana s uchenymi L. Cattabriga, E.L. Roethman, T.D. Djuraeva, S. Abdinazarova, N.N. Shopolova and others. In his work, he developed different methods for the study of linear and nonlinear equations of the third order with multiple characteristics.

Tak kak teoriya uravneniya nechetnogo ryadka eshchyo na stadiya razvitiya rassmatrivaemaya nelokalnye kraevye zadachi dlya vyshenazvannykh uravneniy v nashey dissertatsii slujit dlya razvitii teorii uravneniya tretego ryadka.

In the area of $\Omega = \{(x,t): 0 < x < 1,\ 0 < t \le T\}$ the third-order equations with multiple characteristics are considered

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial t} = 0 \tag{1}$$

with non-local boundary conditions.

It is known that in 1916 the Italian mathematician E.Del Vecchio developed a method for constructing fundamental solutions for equations with multiple characteristics (see [1]).

Then, in 1961, the Italian mathematician L.Cattabriga using the method of E.Del Vecchio constructed the fundamental solutions of equation (1) (see [2])

$$U(x-\xi;t-\tau) = (t-\tau)^{-1/3} f\left(\frac{x-\xi}{(t-\tau)^{1/3}}\right), \quad x \ne \xi, \quad t > \tau \tag{2}$$

$$V(x-\xi;t-\tau) = (t-\tau)^{-1/3} \varphi\left(\frac{x-\xi}{(t-\tau)^{1/3}}\right), \quad x > \xi, \quad t > \tau. \tag{3}$$

Here

$$f(z)=\int_0^\infty \cos\left(\lambda^3-\lambda z\right)d\lambda,\quad -\infty<z<\infty,$$

$$\varphi(z)=\int_0^\infty (\exp\left(-\lambda^3-\lambda z\right)+\sin\left(\lambda^3-\lambda z\right))d\lambda,\quad z>0,$$

$$z=(x-\xi)(t-\tau)^{-1/3}.$$

For the function $f(z),\ \varphi(z)$, which are called Airy functions, the following relations are valid

$$f''(z)+\frac{1}{3}zf(z)=0,\quad \varphi''(z)+\frac{1}{3}z\varphi(z)=0, \tag{4}$$

$$\int_{-\infty}^{\infty} f(z)=\pi,\quad \int_{-\infty}^{0} f(z)=\frac{\pi}{3},\quad \int_0^\infty f(z)=\frac{2\pi}{3},\quad \int_0^\infty \varphi(z)=0, \tag{5}$$

$$f^n(z):\ c_n^+ z^{\frac{2n-1}{4}}\sin\left(\frac{2}{3}z^{3/2}\right),\quad z\to\infty, \tag{6}$$

$$\varphi^n(z):\ c_n^+ z^{\frac{2n-1}{4}}\sin\left(\frac{2}{3}z^{3/2}\right),\quad z\to\infty, \tag{7}$$

$$f^n(z):\ c_n^- |z|^{\frac{2n-1}{4}}\exp\left(-\frac{2}{3}|z|^{3/2}\right),\quad z\to-\infty, \tag{8}$$

Using these properties of the function $f(z),\ \varphi(z)$ L.Cattabriga constructed potential theories for equation (1) (see [2]). He proved that for the function $U(x-\xi;t-\tau),\quad V(x-\xi;t-\tau)$ the following relations are valid

$$\lim_{(x,t)\to(a-0,t)}\int_0^t U_{\xi\xi}(x-a;t-\tau)\alpha(\xi,\tau)d\tau=\frac{\pi}{3}\alpha(t), \tag{9}$$

$$\lim_{(x,t)\to(a+0,t)}\int_0^t U_{\xi\xi}(x-a;t-\tau)\alpha(\xi,\tau)d\tau=-\frac{2\pi}{3}\alpha(t), \tag{10}$$

$$\lim_{(x,t)\to(a+0,t)}\int_0^t V_{\xi\xi}(x-a;t-\tau)\alpha(\xi,\tau)d\tau=0 \tag{11}$$

Further, a number of boundary value problems for equation (1) with local and non-local boundary conditions were considered (see [2]-[5]).

In this section, two non-local problems for equation (1) are considered

Task I. It is required to find a function that is a regular solution of equation (1) in the domain Ω and satisfies the conditions

$$u(x,0)=\mu u(x,T),\quad \mu=const, \tag{12}$$

$$u_{xx}(0,t)=\varphi_1(t),\quad u_x(0,t)=\varphi_2(t),\quad u_x(1,t)=\psi(t). \tag{13}$$

Здесь $K_u=\{u(x,t):\ u(x,t)\in C_{x,t}^{4,1}(\Omega)\cap C_{x,t}^{2,0}(\overline{\Omega}),\quad u_{xt}\in C(\overline{\Omega})\}$.

Theorem 1. Let $0<\mu^2\le\exp\{-T\}$. *Then the task (1), (12), (13) does not have more than one solution.*

Proof. Let the task (1), (12), (13) has two solutions: $u_1(x,t)$, $u_2(x,t)$. Then assuming $v(x,t)=u_1(x,t)-u_2(x,t)$ we get the following problem with respect to the function $v(x,t)$

$$Lv\equiv\frac{\partial^3 v}{\partial x^3}-\frac{\partial v}{\partial t}=0. \tag{14}$$

$$v(x,0)=\mu v(x,T), \tag{15}$$

$$v_{xx}(0,t)=0,\quad v_x(0,t)=0,\quad v_x(1,t)=0. \tag{16}$$

We differentiate equation (14) by x and enter the designation $w(x,t)=v_x(x,t)$. Twhen relative to the function $w(x,t)$ we get the following problem

$$Lw\equiv\frac{\partial^3 w}{\partial x^3}-\frac{\partial w}{\partial t}=0. \tag{17}$$

$$w(x,0)=\mu w(x,T), \tag{18}$$

$$w_x(0,t)=0,\quad w(0,t)=0,\quad w(1,t)=0. \tag{19}$$

Consider the identity

$$\int_0^1\int_0^T L(w)w\exp\{-t\}dxdt=0. \tag{20}$$

Integrating in parts, taking into account homogeneous boundary conditions (18), (19), we have

$$-\frac{1}{2}\int_0^1\int_0^T w^2(x,t)\exp\{-t\}dxdt-\frac{1}{2}\int_0^T w_x^2(1,t)\exp\{-t\}dt-$$

$$-\frac{1}{2}\int_0^1 w^2(x,T)\{\exp\{-T\}-\mu^2\}dx=0.$$

From here, $w(x,t)=0$ in Ω. By virtue of continuity $w(x,t)=0$ in $\overline{\Omega}$. Then $v_x(x,t)=0\Rightarrow v(x,t)=p(t)$. Since $v(0,t)=0$, то $v(0,t)=p(t)=0$ for $\forall t\in[0,T]$. Therefore $v(x,t)=0$ in $\overline{\Omega}$.

Theorem 2. *Let* $\psi(t)\in C^1([0,T])$, $\varphi_2(t)\in C^1([0,T])$, $\varphi_1(t)\in C^1([0,T])$. *Then there is a solution to the problem* (1), (12), (13).

Proof. To prove Theorem 2, we first consider the following auxiliary problem: it is required to find a regular in the domain Ω decision $u(x,t)\in K_u$ equations (1) satisfying the boundary conditions (13) and the initial conditions

$$u(x,0)=\tau(x), \tag{21}$$

Let's construct Green's functions for the problem (1), (13) and (21).

There is an identity

$$\phi L(\theta) - \theta M(\phi) = \frac{\partial}{\partial \xi}\left(\phi\theta_{\xi\xi} - \phi_{\xi}\theta_{\xi} + \phi_{\xi\xi}\theta\right) - \frac{\partial}{\partial \tau}(\phi\theta), \quad (22)$$

где $M(\theta) \equiv -\frac{\partial^3\theta}{\partial \xi^3} + \frac{\partial \theta}{\partial \tau} = 0,\ M -$ operator paired with the operator L, and the functions ϕ and $\theta -$ fairly smooth functions.

Integrating the identity (22) over the domain $\Omega = \{(\xi,\tau): 0 < \xi < 1,\ 0 < \tau \le t\}$, we get

$$\iint_{\Omega}\left[\phi L(\theta) - \theta M(\phi)\right] d\varsigma d\tau = \iint_{\Gamma}\left(\phi\theta_{\xi\xi} - \phi_{\xi}\theta_{\xi} + \phi_{\xi\xi}\theta\right) d\tau + (\phi\theta) d\xi, \quad (23)$$

where the integral in the right part (23) is taken over the entire boundary of the domain Ω.

Now in formula 23 for the functions ϕ and θ let 's take the functions respectively $u(x,t)$ (any regular solution of equation (1)) and $U(x-\xi; t-\tau)$.

Let $\Omega^{\varepsilon} = \{(\xi,\tau): 0 < \xi < 1,\ 0 < \tau \le t - \varepsilon\}$, where $\varepsilon > 0 -$ a fairly small number. Then the identity takes the following form

$$0 = \int_0^{t-\varepsilon}\left[\left(u_{\xi\xi}U - u_{\xi}U_{\xi} + uU_{\xi\xi}\right)\Big|_{\xi=1} - \left(u_{\xi\xi}U - u_{\xi}U_{\xi} + uU_{\xi\xi}\right)\Big|_{\xi=0}\right] d\tau +$$

$$+ \int_0^1 uU\big|_{\tau=0} d\xi - \int_0^1 uU\big|_{\tau=t-\varepsilon} d\xi. \quad (24)$$

From this expression the last integral,

$$\int_0^1 U\left(x-\xi; t-(t-\varepsilon)\right) u(\xi, t-\varepsilon) d\xi = \int_0^1 \varepsilon^{-\frac{1}{3}} f\left(\frac{x-\xi}{\varepsilon^{\frac{1}{3}}}\right) u(\xi, t-\varepsilon) d\xi,$$

consider separately. Now, assuming $z = \frac{x-\xi}{\varepsilon^{\frac{1}{3}}}$, we have

$$\int_{\frac{x-1}{\varepsilon^{1/3}}}^{\frac{x}{\varepsilon^{1/3}}} f(z) u(x - z\varepsilon^{\frac{1}{3}}, t-\varepsilon) dz.$$

Hence in force (5)

$$\lim_{\varepsilon \to 0} \int_{\frac{x-1}{\varepsilon^{1/3}}}^{\frac{x}{\varepsilon^{1/3}}} f(z) u(x - z\varepsilon^{\frac{1}{3}}, t-\varepsilon) dz = \pi u(x,t). \quad (25)$$

Now in (24), aiming at zero and taking into account (25), we get

$$\pi u(x,t) = \int_0^t\left[\left(u_{\xi\xi}U - u_{\xi}U_{\xi} + uU_{\xi\xi}\right)\Big|_{\xi=1} - \left(u_{\xi\xi}U - u_{\xi}U_{\xi} + uU_{\xi\xi}\right)\Big|_{\xi=0}\right] d\tau +$$

$$+\int_0^1 uU\big|_{\tau=0}\,d\xi\,. \qquad (26)$$

Let now $W\left(x-\xi;t-\tau\right)-$ any regular solution of the equation

$$M(\theta)\equiv-\frac{\partial^3\theta}{\partial\xi^3}+\frac{\partial\theta}{\partial\tau}=0\,, \qquad (27)$$

but $u(x,t)-$ any regular solution of equation (1). Then, assuming in the formula (24) $\phi=W,\;\;\theta=u$ we have

$$0=\int_0^t\left[\left(u_{\xi\xi}W-u_\xi W_\xi+uW_{\xi\xi}\right)\Big|_{\xi=1}-\left(u_{\xi\xi}W-u_\xi W_\xi+uW_{\xi\xi}\right)\Big|_{\xi=0}\right]d\tau+$$

$$+\int_0^1 uW\big|_{\tau=0}\,d\xi-\int_0^1 uW\big|_{\tau=t}\,d\xi\,. \qquad (28)$$

From the equalities (26) and (28) we find

$$\pi u(x,t)=\int_0^t\left(u_{\xi\xi}\left(U-W\right)-u_\xi\left(U-W\right)_\xi+u\left(U-W\right)_{\xi\xi}\right)\Big|_{\xi=1}d\tau-$$

$$-\int_0^t\left(u_{\xi\xi}\left(U-W\right)-u_\xi\left(U-W\right)_\xi+u\left(U-W\right)_{\xi\xi}\right)\Big|_{\xi=0}d\tau+$$

$$+\int_0^1 u\left(U-W\right)\Big|_{\tau=0}d\xi+\int_0^1 uW\big|_{\tau=t}\,d\xi\,. \qquad (29)$$

If the regular solution $W(x-\xi;t-\tau)$ equation (27) satisfies the conditions

$$U_{\xi\xi}\,|_{\xi=1}=W_{\xi\xi}\,|_{\xi=1},\qquad U\,|_{\xi=1}=W\,|_{\xi=1},\qquad U_{\xi\xi}\,|_{\xi=0}=W_{\xi\xi}\,|_{\xi=0},\quad W\,|_{\tau=t}=0, \quad (30)$$

then from (29) we will have

$$\pi u(x,t)=-\int_0^t G_\xi(x-1;t-\tau)\psi(\tau)d\tau-$$

$$-\int_0^t G(x-0;t-\tau)\varphi_1(\tau)d\tau+\int_0^T G_\xi(x-0;t-\tau)\varphi_2(\tau)d\tau+$$

$$+\int_0^1 G(x-\xi;t-0)\tau(\xi)d\xi, \qquad (31)$$

where

$$G(x-\xi;t-\tau)=U(x-\xi;t-\tau)-W(x-\xi;t-\tau),$$

Formula (31) gives a solution to the problem (1), (12), (13). However, for this we need to prove the existence of the function $W(x-\xi;t-\tau)$, satisfying equation (27) and conditions (30).

We are looking for the solution of problem (27), (30) in the following form

$$W(x-\xi;t-\tau)=\int_\tau^t U(1-\xi;\eta-\tau)\alpha_1(x,t;\eta)d\eta+$$

$$+\int_{\tau}^{t}U(0-\xi;\eta-\tau)\alpha_2(x,t;\eta)d\eta+\int_{\tau}^{t}V(1-\xi;\eta-\tau)\alpha_3(x,t;\eta)d\eta. \quad (32)$$

Here are the functions $U(x-\xi;t-\tau)$, $V(x-\xi;t-\tau)$ defined by the formula (2), (3), a $\alpha_i(x,t;\eta)$, $i=\overline{1,3}-$ unknown functions yet.

Satisfying the boundary conditions (30) and taking into account the relations (9)-(11) of (32) we have

$$U_{\xi\xi}(x-1;t-\tau)=\frac{\pi}{3}\alpha_1\left(x,t;\tau\right)+\int_{\tau}^{t}U_{\xi\xi}(0-1;\eta-\tau)\alpha_2(x,t;\eta)d\eta, \quad (33)$$

$$U(x-1;t-\tau)=\int_{\tau}^{t}U(1-1;\eta-\tau)\alpha_1(x,t;\eta)d\eta+$$

$$+\int_{\tau}^{t}U(0-1;\eta-\tau)\alpha_2(x,t;\eta)d\eta+\int_{\tau}^{t}V(1-1;\eta-\tau)\alpha_3(x,t;\eta)d\eta, \quad (34)$$

$$U_{\xi\xi}(x-0;t-\tau)=\int_{\tau}^{t}U_{\xi\xi}(1-0;\eta-\tau)\alpha_1(x,t;\eta)d\eta+$$

$$-\frac{2\pi}{3}\alpha_2(x,t;\tau)+\int_{\tau}^{t}V_{\xi\xi}(1-0;\eta-\tau)\alpha_3(x,t;\eta)d\eta. \quad (35)$$

To obtain a system of Volterra integral equations of the second kind, we study expression (34).

Then if we take into account (2), (3), then expression (34) will take the following form

$$U(x-1;t-\tau)=f(0)\int_{\tau}^{t}\frac{1}{(\eta-\tau)^{\frac{1}{3}}}\alpha_1(x,t;\eta)d\eta+$$

$$+\int_{\tau}^{t}\frac{1}{\left(\eta-\tau\right)^{\frac{1}{3}}}f\left(-\frac{1}{\left(\eta-\tau\right)^{\frac{1}{3}}}\right)\alpha_2(x,t;\eta)d\eta+\varphi(0)\int_{\tau}^{t}\frac{1}{\left(\eta-\tau\right)^{\frac{1}{3}}}\alpha_3(x,t;\eta)d\eta.$$

Now we apply the Abel transformation, i.e. first multiplying by $\left(z-\eta\right)^{-\frac{2}{3}}$, then integrating from 0 to z, we get

$$\int_{0}^{z}\frac{1}{\left(z-\tau\right)^{\frac{2}{3}}}U(x-1;t-\tau)d\tau=f(0)\int_{0}^{z}d\tau\int_{\tau}^{t}\frac{1}{\left(z-\tau\right)^{\frac{2}{3}}(\eta-\tau)^{\frac{1}{3}}}\alpha_1(x,t;\eta)d\eta+$$

$$+\int_{0}^{z}d\tau\int_{\tau}^{t}\frac{1}{\left(z-\tau\right)^{\frac{2}{3}}\left(\eta-\tau\right)^{\frac{1}{3}}}f\left(-\frac{1}{\left(\eta-\tau\right)^{\frac{1}{3}}}\right)\alpha_2(x,t;\eta)d\eta+$$

$$+\varphi(0)\int_{0}^{z}d\tau\int_{\tau}^{t}\frac{1}{\left(z-\tau\right)^{\frac{2}{3}}\left(\eta-\tau\right)^{\frac{1}{3}}}\alpha_3(x,t;\eta)d\eta.$$

Now differentiating by we will have

$$\int_0^z \frac{1}{(z-\tau)^{\frac{2}{3}}} U_\tau(x-1;t-\tau)d\tau = -\frac{\pi}{\sqrt{3}} f(0)\alpha_1(x,t;\tau) +$$

$$+\int_0^z d\tau \int_\tau^t \frac{1}{(z-\tau)^{\frac{2}{3}}} U_\tau(-1;\eta-\tau)\alpha_2(x,t;\eta)d\eta - \frac{\pi}{\sqrt{3}}\varphi(0)\alpha_3(x,t;\tau), \quad (36)$$

where

$$B\left(\frac{2}{3},\frac{1}{3}\right) = \int_0^1 \frac{1}{(z-t)^{\frac{2}{3}}(t-\tau)^{\frac{1}{3}}} dt = \frac{\pi}{\sqrt{3}}.$$

We have obtained a system of Volterra integral equations of the second kind (33), (35), (36). From the properties of functions $U(x-\xi;t-\tau),\ V(x-\xi;t-\tau)$ it follows that the solution of the system $\alpha_1(x,t;\tau)\in C(\Omega),\ \alpha_2(x,t;\tau)\in L_1(\Omega),\ \alpha_3(x,t;\tau)\in C(\Omega)$.

Note that for the function $G(x-\xi;t-\tau)$ the same properties that the Green function constructed in [3] for the Cattabriga problem has are valid.

Denote $u(x,T)=\alpha(x)$. Then going to the limit $t\to T$ from (31) we get

$$\pi\alpha(x) = -\int_0^t G_\xi(x-1;T-\tau)\psi(\tau)d\tau -$$

$$-\int_0^t G(x-0;T-\tau)\varphi_1(\tau)d\tau + \int_0^T G_\xi(x-0;T-\tau)\varphi_2(\tau)d\tau +$$

$$+\mu\int_0^1 \{G(x-\xi;T-0)\alpha(\xi)d\xi, \quad (37)$$

So we have obtained an integral Fredholm type equation with respect to the function $\alpha(x)$

$$\alpha(x) = \int_0^1 K(x,\xi)\alpha(\xi)d\xi + F(x), \quad (38)$$

where

$$\mu G(x-\xi;T-0) \equiv K(x,\xi),$$

$$-\int_0^t G_\xi(x-1;T-\tau)\psi(\tau)d\tau - \int_0^t G(x-0;T-\tau)\varphi_1(\tau)d\tau +$$

$$+\int_0^T G_\xi(x-0;T-\tau)\varphi_2(\tau)d\tau \equiv F(x).$$

By virtue of (6)-(9) for the function $K(x,\xi),\ F(x)$ the following relations are valid

$$|K(x,\xi)| < \frac{C}{|x-\xi|^{1/4}},\ F(x)\in C^3([0,1]).$$

By virtue of the uniqueness of the solution of the problem (1), (12), (13) the integral equation (38) has a unique solution.

Task II. Need to find a function $u(x,t) \in D_u$, which is a regular solution of equation (1) in the domain Ω and satisfies the conditions

$$u(x,0) = \mu u(x,T), \qquad \mu = const, \tag{39}$$

$$u(0,t) = \varphi_1(t), \quad u_x(0,t) = \varphi_2(t), \quad u_x(1,t) = \psi(t). \tag{40}$$

Here $D_u = \{u(x,t) : u(x,t) \in C^{3,1}_{x,t}(\Omega) \cap C^{2,0}_{x,t}(\overline{\Omega}),\ u_{xt} \in C(\Omega)\}$.

Theorem 3. *Let* $0 < \mu^2 \le \exp\{-T\}$. *Then the task (1), (39), (40) does not have more than one solution.*

Proof. Let the task (1), (39), (40) has two solutions: $u_1(x,t),\ u_2(x,t)$. Then assuming $v(x,t) = u_1(x,t) - u_2(x,t)$ we get the following problem with respect to the function $v(x,t)$

$$Lv \equiv \frac{\partial^3 v}{\partial x^3} - \frac{\partial v}{\partial t} = 0. \tag{41}$$

$$v(x,0) = \mu v(x,T), \tag{42}$$

$$v(0,t) = 0, \quad v_x(0,t) = 0, \quad v_x(1,t) = 0. \tag{43}$$

Consider the identity

$$\int_0^1 \int_0^T L(v) v_{xt} \exp\{-t\} dxdt = 0. \tag{44}$$

Integrating in parts, taking into account homogeneous boundary conditions (42), (43), we have

$$-\frac{1}{2}\int_0^1\int_0^T v_{xt}^2(x,t)\exp\{-t\}dxdt - \frac{1}{2}\int_0^T v_x^2(1,t)\exp\{-t\}dt -$$

$$-\frac{1}{2}\int_0^1 v_{xx}^2(x,T)\{\exp\{-T\} - \mu^2\}dx = 0$$

From here, $v_{xx}(x,t) = 0$ в Ω, $v_{xx}(x,T) = 0$ в $x \in [0,1]$, $v_t(1,t) = 0$ в $t \in [0,T]$.

Let $\mu^2 < \exp\{-T\}$. Then from these inputs we get: $v_{xx}(x,T) = 0 \Rightarrow v_x(x,T) = const$.

Since $v_x(0,t) = v_x(1,t) = 0 \Rightarrow v_x(0,0) = v_x(0,T) = 0$, то $v_x(x,T) = const = 0$ by $\forall x \in [0,1]$.

Next, we have $v_x(x,T) = 0 \Rightarrow v(x,T) = const \Rightarrow v(x,0) = const$. Since $v(0,t) = 0 \Rightarrow v(0,0) = 0$, that $v(x,0) = const = 0$ by $\forall x \in [0,1]$. Due to the fact that $v_t(1,t) = 0 \Rightarrow v(1,t) = const$ and $v(0,0) = 0$ we have $v(1,t) = 0$.

Then we get the following boundary value problem with respect to the function $v(x,t)$

$$Lv \equiv \frac{\partial^3 v}{\partial x^3} - \frac{\partial v}{\partial t} = 0.$$

$$v(x,0) = 0, \quad v(0,t) = 0, \quad v_x(0,t) = 0, \quad v(1,t) = 0.$$

Due to the work [2], this problem has a unique solution.

Now let $\mu^2 = \exp\{-T\}$. Then $v_{xx} = 0 \Rightarrow v_x(x,t) = \delta_1(t)$ by $\forall t \in [0,T]$. Since $v_x(0,t) = v_x(1,t) = 0$, by $\forall t \in [0,T]$, that $\delta_1(t) = 0$ by $\forall t \in [0,T]$.

Further, $v_x = 0 \Rightarrow v(x,t) = \delta_2(t)$ by $\forall t \in [0,T]$. Since $v(0,t) = 0$, by $\forall t \in [0,T]$, that $\delta_2(t) = 0$ by $\forall t \in [0,T]$.

Then by virtue of continuity $v(x,t) = 0$ in $\overline{\Omega}$.

Theorem 4. *Let* $\psi(t) \in C^1([0,T])$, $\varphi_2(t) \in C^1([0,T])$, $\varphi_1(t) \in C^2([0,T])$. *Then there is a solution to the problem (1), (39),(40).*

Proof. Consider the following auxiliary task:

Find a function $u(x,t) \in D_u$, which is a regular solution of the equation

$$Lu \equiv \frac{\partial^3 u}{\partial x^3} - \frac{\partial u}{\partial t} = 0. \tag{45}$$

in the area of Ω and satisfies the conditions

$$u(x,0) = \tau(x), \tag{46}$$

$$u(0,t) = \varphi_1(t), \quad u_x(0,t) = \varphi_2(t), \quad u(1,t) = \psi(t). \tag{47}$$

Reasoning in exactly the same way as in problem I we get

$$\pi u(x,t) = \int_0^t \left(u_{\xi\xi}(U-W) - u_\xi (U-W)_\xi + u(U-W)_{\xi\xi} \right)\Big|_{\xi=1} d\tau -$$

$$-\int_0^t \left(u_{\xi\xi}(U-W) - u_\xi (U-W)_\xi + u(U-W)_{\xi\xi} \right)\Big|_{\xi=0} d\tau +$$

$$+\int_0^1 u(U-W)\Big|_{\tau=0} d\xi + \int_0^1 uW\Big|_{\tau=t} d\xi. \tag{48}$$

If the regular solution $W(x-\xi;t-\tau)$ equation (27) satisfies the conditions

$$U_{\xi\xi}|_{\xi=1} = W_{\xi\xi}|_{\xi=1}, \quad U|_{\xi=1} = W|_{\xi=1}, \quad U|_{\xi=0} = W|_{\xi=0}, \quad W|_{\tau=t} = 0, \tag{49}$$

then from (48) we will have

$$\pi u(x,t) = -\int_0^t G_\xi(x-1;t-\tau)\psi(\tau)d\tau -$$

$$-\int_0^t G_{\xi\xi}(x-0;t-\tau)\varphi_1(\tau)d\tau + \int_0^T G_\xi(x-0;t-\tau)\varphi_2(\tau)d\tau +$$

$$+\int_0^1 G(x-\xi;t-0)\tau(\xi)d\xi, \tag{50}$$

where

$$G(x-\xi;t-\tau) = U(x-\xi;t-\tau) - W(x-\xi;t-\tau),$$

Formula (50) gives a solution to the problem (1), (39), (40). However, for this we need to prove the existence of the function $W(x-\xi;t-\tau)$, satisfying equation (27) and conditions (49).

We are looking for the solution of problem (27), (49) in the following form

$$W(x-\xi;t-\tau)=\int_{\tau}^{t}U(1-\xi;\eta-\tau)\alpha_1(x,t;\eta)d\eta+$$

$$+\int_{\tau}^{t}U(0-\xi;\eta-\tau)\alpha_2(x,t;\eta)d\eta+\int_{\tau}^{t}V(1-\xi;\eta-\tau)\alpha_3(x,t;\eta)d\eta. \qquad (51)$$

Here are the functions $U(x-\xi;t-\tau),\ V(x-\xi;t-\tau)$ determined by the formula (2), (3), a $\alpha_i(x,t;\eta),\ i=\overline{1,3}-$ as yet unknown functions.

Satisfying the boundary conditions (49) and taking into account the relations (9)-(11) of (51) we have

$$U_{\xi\xi}(x-1;t-\tau)=\frac{\pi}{3}\alpha_1(x,t;\tau)+\int_{\tau}^{t}U_{\xi\xi}(0-1;\eta-\tau)\alpha_2(x,t;\eta)d\eta, \qquad (52)$$

$$U(x-1;t-\tau)=\int_{\tau}^{t}U(1-1;\eta-\tau)\alpha_1(x,t;\eta)d\eta+$$

$$+\int_{\tau}^{t}U(0-1;\eta-\tau)\alpha_2(x,t;\eta)d\eta+\int_{\tau}^{t}V(1-1;\eta-\tau)\alpha_3(x,t;\eta)d\eta, \qquad (53)$$

$$U(x-0;t-\tau)=\int_{\tau}^{t}U(1-0;\eta-\tau)\alpha_1(x,t;\eta)d\eta+$$

$$+\int_{\tau}^{t}U(0-0;\eta-\tau)\alpha_2(x,t;\eta)d\eta+\int_{\tau}^{t}V(1-0;\eta-\tau)\alpha_3(x,t;\eta)d\eta \qquad (54)$$

To obtain a system of Volterra integral equations of the second kind, we study expressions (53) and (54).

Then if we take into account (2), (3), then expression (53) will take the following form

$$U(x-1;t-\tau)=f(0)\int_{\tau}^{t}\frac{1}{(\eta-\tau)^{\frac{1}{3}}}\alpha_1(x,t;\eta)d\eta+$$

$$+\int_{\tau}^{t}\frac{1}{(\eta-\tau)^{\frac{1}{3}}}f\left(-\frac{1}{(\eta-\tau)^{\frac{1}{3}}}\right)\alpha_2(x,t;\eta)d\eta+\varphi(0)\int_{\tau}^{t}\frac{1}{(\eta-\tau)^{\frac{1}{3}}}\alpha_3(x,t;\eta)d\eta.$$

Now we apply the Abel transformation to get

$$\int_{0}^{z}\frac{1}{(z-\tau)^{\frac{2}{3}}}U_{\tau}(x-1;t-\tau)d\tau=-\frac{\pi}{\sqrt{3}}f(0)\alpha_1(x,t;\tau)+$$

$$+\int_{0}^{z}d\tau\int_{\tau}^{t}\frac{1}{(z-\tau)^{\frac{2}{3}}}U_{\tau}(-1;\eta-\tau)\alpha_2(x,t;\eta)d\eta-\frac{\pi}{\sqrt{3}}\varphi(0)\alpha_3(x,t;\tau). \quad (55)$$

Similarly, the expression (54) will be rewritten in the following form

$$U(x-0;t-\tau)=\int_\tau^t \frac{1}{(\eta-\tau)^{\frac{1}{3}}} f\left(\frac{1}{(\eta-\tau)^{\frac{1}{3}}}\right)\alpha_1(x,t;\eta)d\eta+$$

$$+f(0)\int_\tau^t \frac{1}{\left(\eta-\tau\right)^{\frac{1}{3}}}\alpha_2(x,t;\eta)d\eta+\int_\tau^t \frac{1}{\left(\eta-\tau\right)^{\frac{1}{3}}}\varphi\left(\frac{1}{(\eta-\tau)^{\frac{1}{3}}}\right)\alpha_3(x,t;\eta)d\eta.$$

Now we apply the Abel transformation to get

$$\int_0^z \frac{1}{\left(z-\tau\right)^{\frac{2}{3}}}U_\tau(x-0;t-\tau)d\tau=-\int_0^z d\tau\int_\tau^t \frac{1}{\left(z-\tau\right)^{\frac{2}{3}}}U_\tau(1;\eta-\tau)\alpha_1(x,t;\eta)d\eta+$$

$$-\int_0^z d\tau\int_\tau^t \frac{1}{\left(z-\tau\right)^{\frac{2}{3}}}V_\tau(1;\eta-\tau)\alpha_3(x,t;\eta)d\eta+\frac{\pi}{\sqrt{3}}f(0)\alpha_2(x,t;\tau). \quad (56)$$

So, we have obtained a system of Volterra integral equations of the second kind (33), (35), (36). From the properties of functions $U(x-\xi;t-\tau)$, $V(x-\xi;t-\tau)$ it follows that the solution of the system $\alpha_1(x,t;\tau)\in C(\Omega)$, $\alpha_2(x,t;\tau)\in L_1(\Omega)$, $\alpha_3(x,t;\tau)\in C(\Omega)$.

Note that for the function $G(x-\xi;t-\tau)$ the same properties that the Green function constructed in [3] for the Cattabriga problem has are valid.

Denote $u(x,T)=\alpha(x)$. Then going to the limit $t\to T$ from (50) we get

$$\pi\alpha(x)=-\int_0^t G_\xi(x-1;T-\tau)\psi(\tau)d\tau-$$

$$-\int_0^t G_{\xi\xi}(x-0;T-\tau)\varphi_1(\tau)d\tau+\int_0^T G_\xi(x-0;T-\tau)\varphi_2(\tau)d\tau+$$

$$+\mu\int_0^1\{G(x-\xi;T-0)\alpha(\xi)d\xi, \quad (57)$$

So we have obtained an integral Fredholm type equation with respect to the function $\alpha(x)$

$$\alpha(x)=\int_0^1 K(x,\xi)\alpha(\xi)d\xi+F(x), \quad (58)$$

where

$$\mu G(x-\xi;T-0)\equiv K(x,\xi),$$

$$-\int_0^t G_\xi(x-1;T-\tau)\psi(\tau)d\tau-\int_0^t G_{\xi\xi}(x-0;T-\tau)\varphi_1(\tau)d\tau+$$

$$+\int_0^T G_\xi(x-0;T-\tau)\varphi_2(\tau)d\tau\equiv F(x).$$

For the function $K(x,\xi)$, $F(x)$ the following relations are valid

$$|K(x,\xi)|<\frac{C}{|x-\xi|^{1/4}},\ \ F(x)\in C^3([0,1]).$$

By virtue of the uniqueness of the solution of the problem (1), (39), (40) the integral equation (58) has a unique solution.

REFERENCES

1. *Del Vecchio E.* Sulle equazioni $z_{xxx}-z_y+\varphi_1(x,y)=0$, $z_{xxx}-z_{yy}+\varphi_2(x,y)=0$. Memorie R. Accad. Sci. Torino (2). 66 (1915).

2. *Cattabriga L.* Un problema al per una equazione parabolica di ordine dispari. Annali della Souola Normale Sup. di Pisa a mat. 1959, Vol. 13, 2, p. 163-203.

3. *Абдиназаров С.* Об одном уравнения третьего порядка. Изв.АН УзССР, Сер.физ.мат. наук 1989, 6, с 3-7.

4. *Хашимов А.* Об одной задаче для уравнения смешанного типа с кратными характеристиками. Уз. Мат. Журнал, 1995, 2, с 95-97.

5. *Тургинов А.М., Асқаралиева М.А.* об одной нелокальной задаче для уравнения третьего порядка с нелокальным условиям. IJSSIR, Vol. 11, No. 06. June 2022, с 66-73.