JOURNAL OF AERONAUTICAL MATERIALS
ISSN: 1005-5053

Vol. 43, Issue-01, 2023
pp. 1059-1069

WEYL'S TYPE THEOREMS AND BROWDER'S THEOREM FOR PARANORMAL OPERATOR

Dalia S. Ali
Mathematics Department, College of Science, University of Baghdad, Baghdad, Iraq,
dalia.sami1103a@sc.uobaghdad.edu.iq

Buthainah A.A. Ahmed
Mathematics Department, College of Science, University of Baghdad, Baghdad, Iraq
buthaina.a@sc.uobaghdad.edu.iq

Abstract. In this article we studied some spectral properties for the paranormal operator which is densely defined closed operator. This type of operator can be represents as a matrix representation, by using this representation we can study the Weyl's types theorems and Browder's theorem. In addition, under sufficient and necessary conditions the paper explain that the essential spectrum, the Weyl's spectrum, and the Browder's spectrum of such an operator matrix corresponds to the union of the essential spectrum, the Weyl spectrum and the Browder's spectrum of its diagonal elements.
Keywords. Browder's Theorem, Riesz Projection, Spectral Properties, Unbounded Paranormal Operators, Weyl's Theorem.

1. Introduction

In 1909, H. Weyl tasted the spectrum of all compact perturbations of self-adjoint operators in Hilbert space and found that their intersections consisted of points in a spectrum of finite multiplicity that were not isolated eigenvalues. The bounded linear operators that satisfy this property are said to satisfy Weyl's theorem [1]. Subsequently, Berkani and Weyl introduced some variants of Weyl's theorem, this study is commonly known as the Weyl type's theories including the a-Weyl's theorem. In 2004, researchers presented the Browder's theorem and a-Browder theorem as generalizations of a-Weyl's theorem [2]. Some spectrum properties has been studied for classes of operators that are bounded (see [3], [4]).

One of the most important and well-studied classes in operator theory is that of Normal operators. Let $N_o(H) = \{S: H \rightarrow H \; such \; that \; S \; is \; normal\}$,the spectrum theorem for $S \in N_o(H)$ guarantees the existence of non-trivial invariant subspaces and also reveals the complete structure of operators. Normal operators thus lead to several generalizations and one of these generalizations is the class of paranormal operators. The class of finite paranormal operators was first studied by Istrătescu [5]. Furthermore, Furuta [6] introduced the term paranormal operator. Limited paranormal operators have been studied by many authors. For example: [7, 8].

Weyl's theorem and the self-adjointness of the Riesz projection with respect to the separated spectral values of the operators have been studied for many different classes of operators. This was established by his Coburn [9] for some non-regular operators (non-regular operators and

Toeplitz operators). In addition, Uchiyama [8] used Ando's characterization [10] for paranormal operators and extended it to boundary paranormal operators. However, Ando's characterization is not available for Bounded Paranormal Operators, so Bounded Operator techniques will not work in this case.

In this paper we are using the densely defined closed paranormal operators in infinite dimensional Hilbert space H and prove some Weyl's type theorems and Browder's theorems. Also we proves that these operators are obeys some spectral properties.

2. Notation and Preliminaries

In this section, we recall the following concepts, which are used later.

All through this work, $\mathrm{H}, \mathrm{H}_1, \mathrm{H}_2$ denotes to infinite dimensional complex Hilbert spaces, $C(\mathrm{H})$ is the set of all closed linear operators defined on H. For an operator $S \in C(\mathrm{H})$, we define $N(S)$ as the kernel of S, while $D(S)$ represents the domain, and $R(S)$ denotes the range of S. The upper semi Fredholm operator is define if $R(S)$ is closed and $n(S) = \dim N(S)$ is finite while we say that S is lower semi Fredholm operator if $d(S) = codim\ R(S) = \dim(N(S)^{\perp})$ is finite. A Fredholm operator is upper and lower semi Fredholm operator.

$$SF_{+}(\mathrm{H}) = \{S \in C(\mathrm{H}): S\ is\ upper\ semi\ Fredholm\},$$

$$SF_{-}(\mathrm{H}) = \{S \in C(\mathrm{H}): S\ is\ lower\ semi\ Fredholm\}.$$

The index of S is defined as ind $(S) = n(S) - d(S)$.

An operator $S \in C(\mathrm{H})$ which is Fredholm operator of index 0 is defined as Weyl operator, while $\sigma_w(S) = \{\eta \in \mathbb{C} : S - \eta I$ is not weyl$\}$ is used to define the Weyl spectrum of S. In addition we can assign the following notations:

$SF_{+}^{-}(\mathrm{H}) = \{S \in C(\mathrm{H}):\ S \in SF_{+}(\mathrm{H}), ind(S) \leq 0\ \}$

$SF_{-}^{+}(\mathrm{H}) = \{S \in C(\mathrm{H}): S \in SF_{-}(\mathrm{H}), ind\ (S) \geq 0\}$

In [3], Berkani generalized the concept of Fredholm operators to B-Fredholm operators as follows

$$\Omega(S) = \{i \in \mathbb{N}: \forall j \in \mathbb{N}, j \geq i \Rightarrow R(S^i) \cap N(S) \subseteq R(S^j) \cap N(S)\}$$

The degree of stable iteration of S is denoted by $dis(S)$ and defined by $dis(S) = inf\ \Omega(S)$ and $dis(S) = \infty$ when $\Omega(S) = \emptyset$.

Furthermore, for $S \in C(\mathrm{H})$ the $B-Fredholm$ operator is upper and lower semi $B-Fredholm$ operator, where S is upper (resp.,lower) semi $B - Fredholm$ operator if $\exists\ d \in \Omega(S)$: $\dim\{N(S) \cap R(S^d)\}$ is finite and $R(S^d)$ closed and (resp., $codim\ \{R(S) + N(S^n)\}$ is finite), and the index of S is

$$\text{ind}(S) = dim\{N(S) \cap R(S^d)\} - codim\{R(S) + N(S^d)\}.$$

We call $S \in C(\text{H})$ as $B - W$eyl if it's B- Fredholm operator with $index$ 0 and $\sigma_{\text{BW}}(S)$ is used to symbolize the B-Weyl spectrum of S and defined by $\sigma_{\text{BW}}(S) = \{\eta \in \text{C} : S - \eta \text{ I is not } B\text{-weyl}\}$.

Moreover, the ascent $\text{asc}(S)$ and descent $\text{dsc}(S)$ for $S \in C(\text{H})$ are defined as:

$$\text{asc}(S) = \inf\{d: N(S^d) = N(S^{d+1})\}$$
$$\text{dsc}(S) = \inf\{d: R(S^d) = R(S^{d+1})\}$$

An operator $S \in C(\text{H})$ is called Browder if it's both upper and lower semi Browder, where $S \in C(\text{H})$ is upper semi- Browder if $\text{asc}(S) < \infty$ with S is upper semi- Fredholm and it is lower semi-Browder if $\text{dsc}(S) < \infty$ with S is lower semi – Fredholm.

Now, we can define the following spectrum for an operator S as:

$\sigma_{SF_+}(S) = \{\eta \in \mathbb{C}: S - \eta\text{I not upper semi Fredholm}\}$,

$\sigma_{SF_-}(S) = \{\eta \in \mathbb{C}: S - \eta\text{I not lower semi Fredholm}\}$,

$\sigma_{\text{e}}(S) = \{\eta \in \mathbb{C}: S - \eta\text{I not Fredholm}\}$,

$\sigma_{SF_+^-}(S) = \{\eta \in \mathbb{C}: S - \eta\text{I} \notin SF_+^-(\text{H})\}$,

$\sigma_{\text{ub}}(S) = \{\eta \in \mathbb{C}: S - \eta\text{I not upper semi-Browder}\}$,

$\sigma_{\text{Ib}}(S) = \{\eta \in \mathbb{C}: S - \eta\text{I not lower Semi-Browder}\}$ and

$\sigma_b(S) = \{\eta \in \mathbb{C}: S - \eta\text{I not Browder}\}$, respectively.

Evidently

$$\sigma_e(S) \subset \sigma_w(S) \subset \sigma_b(S) = \sigma_e(S) \cup \text{acc } \sigma(S),$$

where acc $\sigma(S)$ denotes the set of accumulation points of the spectrum $\sigma(S)$ of S.

Recall that one says that S obeys Weyl's theorem if

$$\sigma(S) \setminus \sigma_w(S) = E_0(S),$$

where $E_0(S)$ is the set of isolated points of $\sigma(S)$ which are eigenvalues of finite multiplicity, and that one says that S obeys Browder's theorem if $\sigma_w(S) = \sigma_b(S)$, or

$$\sigma(S) \setminus \sigma_{\text{w}}(S) = \Pi^0(S)$$

Where

$$\Pi(S) = \{\eta \in iso\ \sigma(S): 0 < asc(S-\eta I) = dsc(S-\eta I) < \infty\}$$

$$\Pi^0(S) = \{\eta \in \Pi(S): n(S-\eta I) < \infty\}.$$

We say that S obeys a-Weyl's theorem if

$$\sigma_a(S) \setminus \sigma_{SF_+^-}(S) = E_0^a(S),$$

where $E_0^a(S)$is the set of isolated points of $\sigma_a(S)$ which are eigenvalues of finite multiplicity, and that S obeys a-Browder's theorem if $\sigma_{SF_+^-}(S) = \sigma_{ub}(S)$.

Remark 2.1. If $S \in \mathcal{C}(\mathrm{H}_1, \mathrm{H}_2)$ and $N(S) = \{0\}$, then the inverse operator, S^{-1} is the linear operator from H_2 to H_1, with $D(S^{-1}) = R(S)$ and $S^{-1}(Sx) = x$ for all $x \in D(S)$. In particular if $S \in \mathcal{C}(\mathrm{H})$ is densely defined and bijective, then by the closed graph theorem it follows that $S^{-1} \in \mathcal{B}(H)$ where $\mathcal{B}(H)$ is the set of bounded linear operator defined on H (See [11]).

Remark 2.2. (See [12]) Suppose that closed operator S is a Fredholm operator and K is S-compact. Then $S + K$ is a Fredholm operator and ind $(S + K) =$ ind (S).

Definition 2.3. (See [13]) Let $S \in \mathcal{C}(\mathrm{H})$ with $\sigma(S) = \sigma \cup \tau$, where σ is contained in some bounded domain Ω such that $\bar{\Omega} \cap \tau = \emptyset$. Let Λ be the boundary of Ω, then

$$\mathfrak{R}_\sigma = \frac{1}{2\pi} \int_\Lambda (zI - S)^{-1} dz, \qquad (2.1)$$

is called the Riesz projection with respect to σ.

Theorem 2.4. [14, Theorem 2.1, Page 326] Suppose $S \in \mathcal{C}(H)$ with $(T) = \sigma \cup \varsigma$, where σ is contained in some bounded domain Ω and $\mathfrak{R}_\sigma$ is the operator defined in Equation 2.1. Then

1. $\mathfrak{R}_\sigma$ is a projection.
2. The subspace $R(\mathfrak{R}_\sigma)$ and $N(\mathfrak{R}_\sigma)$ are invariant under S.
3. The subspace $R(\mathfrak{R}_\sigma)$ is contained in $D(\mathrm{S})$ and $S|_{R(\mathfrak{R}_\sigma)}$ is bounded.
4. $\sigma(S|_{R(\mathfrak{R}_\sigma)}) = \sigma$ and $\sigma(S|_{N(\mathfrak{R}_\sigma)}) = \varsigma$.

In particular, if $\eta \in iso\ \sigma(T)$, then there exist a positive real number r such that $\{z \in \mathbb{C}: |z - \eta| \le r\} \cap \sigma(S) = \{\eta\}$. If we take Λ to be the boundary of $\{z \in \mathbb{C}: |z - \eta| \le r\}$, then the Riesz projection with respect to η is defined as

$$\mathfrak{R}_\eta = \frac{1}{2\pi i} \int_\Lambda (zI - S)^{-1} dz \qquad (2.2)$$

Definition 2.5. [7, Definition 1.1] An operator $S \in \mathcal{L}(H)$ is called paranormal operator if

$$\| Sx \|^2 \le \|S^2 x\| \| x \|, \forall x \in D(S^2). \qquad (2.3)$$

Equivalently, S is paranormal, if $\| Sx \|^2 \leq \|S^2x\|, \forall x \in S_{D(S^2)}$.

If $S \in \mathcal{B}(\mathrm{H})$, then Equation 2.3 holds for every $x \in \mathrm{H}$.

It is well known that spectrum of a densely defined closed normal operator is non-empty. The following theorem proves this result for the class of paranormal operators.

Theorem 2.6. [16, Theorem 4.2, Page 12] If $S \in \mathcal{C}(\mathrm{H})$ be a densely paranormal operator, then $\sigma(S) \neq \emptyset$.

It was shown by Istrătescu-Saito-Yoshino [5] that a paranormal operator whose spectrum is contained in the unit circle is always unitary operator. Also it is well-known that the inverse operator of an invertible paranormal operator is always paranormal. The following lemma shows that every isolated point in spectrum of paranormal operator S is eigenvalue.

Lemma 2.7. [15, Proposition 3.7, Page 8] Let S be a paranormal operator and $\eta \in$ iso $\sigma(S)$.Then the Riesz idempotent $\mathfrak{R}$ with respect to η defined by (2.2) satisfies ran $\mathfrak{R} =$ dim$(N\,(S - \eta))$. Hence, η is an eigenvalue of S.

Corollary 2.8. [15, Proposition 3.9, Page 8] Let S be as defined in Lemma 2.4 and $\eta \in iso\ \sigma(S)$. Then $N(\mathfrak{R}_\lambda) = R(S - \eta I)$.

The next lemma gives a characterization of closed range paranormal operators. **Lemma 2.8.** [15, Proposition 3.10, Page 9] Suppose $S \in \mathcal{C}(\mathrm{H})$ is a densely defined paranormal operator. If $0 \in iso\ \sigma(S)$, then $R(S)$ is closed.

3. WEYL'S TYPES THEOREMS AND BROWDE'S THEOREM FOR UNBOUNDED PARANORMAL OPERATOR.

In this section we show that a densely defined closed paranormal operator T satisfy a-Weyl's theorem, Browder's theorem, Generalized Weyl's theorem, Generalized Browder's theorem, Property (b), Property (gw) and Property (gb).

For $S \in \mathcal{C}(H)$, where $H = H_1 \oplus \mathrm{H}_2$ then S has the block matrix representation

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \qquad (3.1)$$

where $S_{ij}: D(S) \cap H_j \to H_i$ is defined by $S_{ij} = P_{H_i} S P_{H_j}\Big|_{D(S)\cap H_j}$ for $i,j = 1,2$. Here P_{H_i} is an orthogonal projection onto H_i.

For $(x_1, x_2) \in (H_1 \cap D(S)) \oplus (H_2 \cap D(S))$,

$$S(x_1, x_2) = (S_{11}x_1 + S_{12}x_2, S_{21}x_1 + S_{22}x_2).$$

Note that if S is densely defined then T_{ij} is densely defined for $i, j = 1,2$, that is $\overline{D(S_{ij} \cap H_j)} = H_j$ for all $i, j = 1,2$. See [15]

Remark 3.1. [15] Let S be as defined in Equation 3.1. If $H_1 = N(S) \neq \{0\}$ and $H_2 = N(S)^{\perp}$, then

$$S = \begin{bmatrix} 0 & S_{12} \\ 0 & S_{22} \end{bmatrix} \qquad (3.2)$$

1. If S is densely defined closed operator then S_{22} is also densely defined closed operator.
2. It can be easily checked that $R(S_{22}) = R(S) \cap N(S)^{\perp}$. If $R(S)$ is closed, then $R(S_{22})$ is closed.

In [9] and [8], Coburn and Uchiyama proved that any bounded hyponormal, Toeplitz and paranormal operator satisfies the Weyl's theorem. In [15] Bala and Ramish Prove this theorem for unbounded paranormal operators. Here we are going to prove more types of Weyl's theorems, Browder's theorem and some spectral properties for *densely defined* Paranormal operator.

Theorem 3.2. If $S \in C(H)$ is *densely defined* Paranormal operator. Then S satisfy a-Weyl's theorem.

Proof. Let $\eta \in \sigma_a(S) \backslash \sigma_{SF_+^-}(S)$. So, $R(S - \eta I)$ is closed and $ind(S - \eta I) \leqslant 0$. If $ind(S - \eta I) < 0$ then $S - \eta I$ can't be decomposed as a matrix representation in Equation 3.2. Assume $ind(S - \eta I) = 0$, then $N(S - \eta I) > 0$ since if $N(S - \eta I) = N(S - \eta I)^* = \{0\}$ then $S - \eta I$ has bounded inverse thus $\eta \notin \sigma(S)$ which is controduction. Then $S - \eta I$ can be decomposed on $H = H_1 = N(S - \eta I) \oplus H_2 = N(S - \eta I)^*$ as

$$S - \eta I = \begin{bmatrix} 0 & S_{12} \\ 0 & S_{22} - \eta I \end{bmatrix}$$

by remark 3.1 , $S_{22} - \eta I$ is densely defined operator with $R(S_{22} - \eta I)^{\perp}$ is closed. Since $N(S - \eta I)$ is finite dimensional subspace then, S_{12} is finite rank operator, so we have

$$\mathrm{ind}(S_{22} - \eta I) = \mathrm{ind}(S - \eta I) = 0$$

Since $N(S_{22} - \eta I)^* = 0$, then $\overline{R(S_{22} - \eta I)} = N(S - \eta I)^{\perp}$thus $\eta \notin \sigma(S_{22})$. It's easy to see $\sigma(S) \subseteq \sigma(S_{22}) \cup \{\eta\}$, so η is an isolated point of $\sigma_a(S)$ hence, $\eta \in E_0^a(S)$.

Conversely, assume $\eta \in E_0^a(S)$, then η is isolated approximate value, with $0 < \dim(N(S - \eta I)) < \infty$. It remains to prove $R(S - \eta I)$ is closed. Consider the general Riez projection $\mathfrak{R}$ with respect to η defined in Equation 2.2. By theorem 2.5 and corollary 2.8. we have

$$\begin{aligned} R(S-\eta I) &= R\left((S-\eta I)|_{N(\mathfrak{R}_\eta)}\right) \\ &= N(\mathfrak{R}_\eta). \end{aligned}$$

Thus, $R(S-\eta I)|_{N(\mathfrak{R}_\eta)}\Big)$ is closed so is $R(S-\eta I)$. This proves our theorem.

Theorem 3.3. If $S \in C$(H) be *densely defined* paranormal operator then $\sigma(S)\backslash\sigma_b(S) \subseteq E(S)$.. Where $E(S) = \{\eta \in iso\ \sigma(S): 0 < n(S-\eta I)\}$

Proof. Let $\eta \in \sigma(S)\backslash\sigma_b(S)$. So we have $n(S-\eta I) = d(S-\eta I) < \infty$ and $R(S-\eta I)$ is closed. By remark 3.1 $S-\eta I$ can be written as

$$S-\eta I = \begin{bmatrix} 0 & S_{12} \\ 0 & S_{22}-\eta I \end{bmatrix}$$

with $H = H_1 \oplus H_2$ where $H_1 = N(S-\eta I)$, $H_2 = N(S-\eta I)^{\perp}$

Hence $S_{22}-\eta I$ is densely define operator with $R(S_{22}-\eta I)$ is closed.

Moreover, since S_{12} is finite rank operator then $ind(S-\eta I) = ind(S_{22}-\eta I)$. Since $N(S_{22}-\eta I) = \{0\}$ by corollary 2.1 $S_{22}-\eta I$ is an invertible operator on $\overline{R(S-\eta I)}$, hence $\eta \notin \sigma(S_{22})$ thus $\sigma(S) \subseteq \{\eta\} \cup \sigma(S_{22})$ S. η is an isolated point of spectrum.

Lemma 3.4. If $S \in C$(H) is *densely defined* operator then the following holds

1. S is upper B-Fredholm operator with $n(S) < \infty$ if and only if S is upper semi Fredholm operator.
2. S is lower B-Fredholm operator with $d(S) < \infty$ if and only if S is lower semi Fredholm operator.

Proof. The proof is similar to the bounded case.

Corollary 3.5. Let $S \in C$(H) be a *densely defined* paranormal operator with $asc(S-\eta I) < \infty$ then S satisfies

1 Generalized Weyl theorem.

2 Generalized Browder's theorem.

3 Property (gw).

4 Property (gb).

Theorem 3.6. If $S \in C$(H) is *densely defined* paranormal operator. Then S satisfies Browder's theorem.

Proof. Let $\eta \in \sigma(S)\backslash\sigma_w(S)$, then $n(S-\eta I) = d(S-\eta I) < \infty$ with $R(S-\eta I)$ is closed.

Then $S-\eta I$ can be written as matrix representation shown in remark 3.1 as

$$S-\eta I=\begin{bmatrix}0 & S_{12}\\ 0 & S_{12}-\eta I\end{bmatrix}$$

By remark 2.1 and since S_{12} is compact operator then $ind(S-\eta I)=ind(S_{22}-\eta I)=0$

Since $N(S_{22}-\eta I)=\{0\}$ then $S_{22}-\eta I$ is invertible operator on $\overline{R(S_{22}-\eta I)}$, hence $\eta\notin\sigma(S_{22})$. Then $\sigma(S)\subseteq\{\eta\}\cup\sigma(S_{22})$, η is isolated point in $\sigma(S)$.

Since S is paranormal operator, then S has SVEP at η. and by $S-\eta I$ is Weyl operator asc $(S-xI)<\infty$ (See [16, theorem 3.8]). Now, by remark 3.4 (iv) of [16] we get $asc(S-\eta I)=dsc(S-\eta I)<\infty$. Hence $\eta\in\prod^0(S)$.

For the other inclusion, let $\eta\in\prod^\circ(S)$, then we get $n(S-\eta I)<\infty$. Consider the Riez Projection $\mathfrak{R}_\eta$ which defined in Equation 2.2. Hence,

$$R(S-\eta I)=N(\mathfrak{R}_\eta),\eta\notin\sigma\left(S|_{N(\mathfrak{R}_\eta)}\right)$$

which implies $R(S-\eta I)$ is closed and $\left.(S-\eta I)|_{N(\mathfrak{R}_\eta)}\right)^{-1}\in B(I)$, thus

$$\begin{aligned}n(S-\eta I)^* &= d(S-\eta I)^\perp\\ &=\dim\left(N(\mathfrak{R}_\eta)^\perp\right)\\ &=\dim\left(R(\mathfrak{R}_\eta)\right)\\ &=n(S-\eta I)\end{aligned}$$

Hence, $S-\eta I$ is Weyl's operator. Thus $\eta\notin\sigma_w(S)$. The proof is complete.

Corollary 3.7. Let $S\in C$(H) be a *densely defined* paranormal operator then S satisfies Weyl's theorem if and only if S satisfies Browder's theorem.

Theorem 3.8. Let $S\in C$(H) be a *densely defined* paranormal operator then S obeys property (b).

Proof. Let $\eta\in\sigma_a(S)\backslash\sigma_{SF_+^-}(S)$. Assume $ind(S-\eta I)=0$, then $\dim(N(S-\eta I))=\dim(N(S-\eta I)^*)<\infty$ with $R(S-\eta I)$ is closed. We can decomposed $S-\eta I$ as

$$\mathrm{S}-\eta I=\begin{bmatrix}0 & S_{12}\\ 0 & S_{22}-\eta I\end{bmatrix}$$

on $H=H_1\oplus H_2$.

since $S_{22} - \eta I$ is invertible bounded operator then $\eta \notin \sigma(S_{22})$. Thus $\sigma(S) \subseteq \{\eta\} \cup \sigma(S_{22})$, hence $\eta \in iso\, \sigma(S)$.

Since S is paranomal and $S - \eta I$ is Semi Fredholm operator then $\mathrm{asc}(S - \eta I) < \infty$ (See [16, theorem 3.8]), by [16, theorem 3.4 (iv)] we get $\eta \in \prod^0(S)$

Conversely, let $\eta \in \prod^0(S)$, by using Riez projection operator $\mathfrak{R}$ on η we have

$$dim(N(S - \eta I)^* = dim\,(\mathfrak{R}_\eta)^* = dim\,(N(S - \eta I)$$

Hence, $R(S - \eta I)$ is closed.

Remark 3.9. Let $S \in C(\mathrm{H})$ be a *densely defined* paranormal operator defined as in remark 3.1 such that

$$\mathrm{S} = \begin{bmatrix} 0 & S_{12} \\ 0 & S_{22} \end{bmatrix}$$

If S_{12} is closable operator then by [12] and [17], we get the following:

1. $\sigma_e(S_{11}) \cup \sigma_e(S_{22}) = \sigma_e(S) \cup \left(\sigma_{p\infty}(S_{22}) \cap \sigma_{p\infty}({S_{11}}^*)^*\right)$

where $\sigma_{p_\infty}(\cdot) = \{\eta \in \sigma_p(\cdot): n(\cdot - \eta I) = \infty\}$ and $\sigma_{p\infty}(\cdot)^* = \{\eta \in \mathbb{C}: \eta \in \sigma_{p\infty}(\cdot)\}$

2.

$$\sigma_w(S_{11}) \cup \sigma_w(S_{22}) = \sigma_w(S) \cup \left(\sigma_{p+}(S_{22}) \cap \sigma_{p+}({S_{11}}^*)^*\right) \cup \left(\sigma_{p+}(S_{11}) \cap \sigma_{p+}({S_{22}}^*)^*\right)$$

where $\sigma_{p+}(\cdot) = \{\eta \in \sigma_p(\cdot): \eta(\cdot - \eta I) > d(\cdot - \eta I)\}$.

3.

$$\sigma_b(S_{11}) \cup \sigma_b(S_{22}) = \sigma_b(T) \cup \sigma_{\mathrm{asc}}(S_{22})$$

where $\sigma_{\mathrm{asc}}(\cdot) = \{\eta \in \mathbb{C}: \mathrm{asc}\,(\cdot - \eta I) = \infty\}$

4.

$$\sigma_b(S) = \sigma_b(S_{11}) \cup \sigma_b(S_{22})$$

if and only if

$$\sigma_{\mathrm{asc}}(S_{22}) \subset \sigma_b(S).$$

In particular, if $\sigma_{\mathrm{asc}}(S_{22}) = \emptyset$, then $\sigma_b(S) = \sigma_b(S_{11}) \cup \sigma_b(S_{22})$.

5.

$$\sigma_{\star}(S) = -\sigma_{\star}({S_{11}}^{*}) \cup \sigma_{\star}(S_{11}),$$

where $\sigma_{\star} \in \{\sigma_e, \sigma_w, \sigma_b\}$.

6.

$$\sigma_{S_{F_+^-}}(S_{11}) \cup \sigma_{SF_+^-}(S_{22}) = \sigma_{SFF_+^-}(S)$$

if and only if

$$\sigma_{p_+}(S_{22}) \cap \sigma_{p_+}({S_{11}}^{*})^{-} \subseteq \sigma_{SF_+^-}(S)$$
$$\text{and } \sigma_{p_+}(S_{11}) \cap \sigma_{p_+}({S_{22}}^{*})^{-} \subseteq \sigma_{SF_+^-}(S),$$
$$\sigma_{p_{p\infty}}({S_{11}}^{*})^{-} \cap \sigma_{p_{\infty\infty}}(S_{22}) \subseteq \sigma_{SF_+^-}(S).$$

In particular, if $\sigma_{p_+}(S_{22}) \cap \sigma_{p_+}(S_{11})^{-} = \emptyset$ and $\sigma_{p_+}(S_{11}) \cap \sigma_{p_+}({S_{22}}^{*})^{-} = \emptyset, \sigma_{p_\infty}({S_{11}}^{*})^{-} \cap \sigma_{p_\infty}(S_{22}) = \emptyset$, then $\sigma_{SF_+^-}(S_{11}) \cup \sigma_{S_+^-}(S_{22}) = \sigma_{\mathrm{SF}_+^-}(S)$.

7.

$$\sigma_{lb}(S_{11}) \cup \sigma_{lb}(S_{22}) = \sigma_{ib}(S)$$

if and only if

$$\sigma_{asc}(S_{22}) \subseteq \sigma_{lb}(S)$$
$$\text{and } \sigma_{p_\infty}({S_{11}}^{*})^{-} \cap \sigma_{p_{\infty\infty}}(S_{22}) \subseteq \sigma_{lb}(S).$$

In particular, if $\sigma_{asc}(S_{22}) = \emptyset$ and $\sigma_{p_\infty}({S_{11}}^{*})^{-} \cap \sigma_{p_{\infty c}}(S_{22}) = \emptyset$, then $\sigma_{lb}(S_{11}) \cup \sigma_{lb}(S_{22}) = \sigma_{lb}(S)$.

8.

$$-\sigma_{*}({S_{11}}^{*}) \cup \sigma_{*}(S_{11}) = \sigma_{*}(S),$$

where $\sigma_{\star} \in \left\{\sigma_{SF_+^-}, \sigma_{lb}\right\}$.

References.

1. Weyl, H.: Über beschränkte quadatische Formen, deren Differenz Vollstetig ist. Rend. Circ. Mat. Palermo, 27 (1909), 373-392.
2. Berkani, M. and Arroud,A.: Generalized Weyl's theorem and hyponormal operators. J. Aust. Math. Soc. 76 (2) (2004), 291-302.
3. Buthainah A.A. Ahmed, Zaman A., New Properties and Their Relation to type Weyl-Browder Theorems, IJSR, 2 (2018), 1502 – 1506.
4. Rashid, Z. A. ., & Ahmed, B. A. . Some Properties of Algebraically Paranormal Operator. Iraqi Journal of Science, 63(8) (2022), 3587–3591.

5. V. Istritescu, On some hyponormal operators, Pacifie 1. Math. 22 (1967), 413-417.
6. T. Furuta, On the class of paranormal operators, Proc. Japan Asad. 49 (1967), 594-598.
7. A. Daniluk, On the closability of paranormal operators, J. Math. Anal. Appl. 976 (2011) ,342-348.
8. A. Uchiysme, On the isolated points of the spectrum of paranormal operators, Integral Equations Operator Theory 55 (2006), 145-151.
9. L. A. Coburn, Weyl's theorem for nonnormal operators, Michigan Math. I. 13 (1966), 264-268.
10. T. Ando, Operators with a norm condition, Acta Sci. Math. (Szeged) 33 (1972), 169-178.
11. S. Goldbarg, Unbounded linear operators: Theory and applications, MeCrnw-Hill Boak Ca., New York, 1966.
12. Qingmei Bai, Junjie Huang, Alatancang Chen, Weyl type theorems of 2×2 upper triangular operator matrices, Journal of Mathematical Analysis and Applications, 434 (2) (2016), 1065-1076.
13. J. V. Baxlay, On the Weyl spectrum of a Hilbert space operators, Proe Amer. Math. Soc. 34 (1972), 447-452.
14. I. Gohberg, S. Goldberg and M. A. Kasshock, Classes of linear operators. Vol. I, Operator Theory: Advances and Applications, 49, Birkhāusor Verlag, Bsecl, 1990.
15. Bala, N., Ramesh, G. Weyl's theorem for paranormal closed operators. Ann. Funct. Anal. 11 (2018), 567–582.
16. P. Aiena, Fredholm and Local Spectral Theory, with Applications to Multipliers, Kluwer Academic Publishers, 2004.
17. Qingmei Bai, Junjie Huang & Alatancang Chen, Essential, Weyl and Browder spectra of unbounded upper triangular operator matrices, Linear and Multilinear Algebra, 64:8 (2018), 1583-1594.