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AN INTRODUCTION TO THE STUDY OF PARTIAL DIFFERENTIAL EQUATIONS USING THE EULER-BERNOULLI MODEL FOR THE TRANSVERSE VIBRATION OF A FLEXIBLE BAR

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Abstract

Differential equations content is part of courses like math, physics and engineering. Since it is an extension of the discipline of differential and integral calculus, it also includes the knowledge of well-established principles around it. This is a commentary on this subject and its effect on the learning of differential equations. This work is a brief introduction to Partial Differential Equations by examining the model of Euler-Bernoulli for a uniform flexible bar cross vibrations. This paper further explains the Euler-Bernoulli modeling for a flexible bar which results in a fourth order partial differential equation to try to present a problem in the proposed learning and outlines the basic concepts of physics and mathematics required to understand the Euler-Bernoulli modeling and solution. The study is made for small displacements or small slopes of the bar, which makes it possible to easily integrate the differential equation of the extruded bar and calculate the elastic and vertical displacement of the free end. The results obtained have been compared with the exact ones as well as with the experimental measurements carried out in the laboratory with a steel ruler embedded in one end the equations of the elastic are obtained following a treatment analogous to that developed by Feynman when he studies the extruded beam in his Physics book, so the development presented here can be followed without many problems by any student of Physics or Mechanics of the first courses university students.

Keywords: partial differential equations, Euler-Bernoulli model, solution, slope, vibration of a flexible bar.

1 Introduction

Differential equations are subjects of intensive activities that require both scientific and technical study precisely because they offer, in addition to rigorous mathematics, a diversity of applications [1]. To better understand the force that lies in the equations, it is important to see their roots and how they have evolved over time. In addition to having evolved for the purpose of explaining phenomena from other sciences, each equation tells a story and contributes to the development of mathematics that we see today [2]. The Euler-Bernoulli equation [3] represents a basic model of bending vibrations of beams when all the simplifying assumptions are met. The Euler-Bernoulli theory for beams, sometimes called, classical thin beam theory, Euler beam theory [4], Bernoulli beam theory or Bernoulli-Euler beam theory, remains in its simplicity, and thanks to the reasonable results that it provides, the theory most used in the approximation of the vibrations of beams. However, the Euler or Euler-Bernoulli model tends to slightly overestimate the natural frequencies, a problem which becomes important for the frequencies of higher order modes or when the thickness of the beam is important. Different characteristics of the thin bar embedded in one end will be obtained: the maximum horizontal and vertical displacements of the free end of the bar as a function of the applied force, and the curve that the bent bar adopts and is called elastic [5]. Both the maximum and the elastic displacements depend on the material and the geometry of the bar. Some numerical results will be shown and they will be compared with the results obtained using the approximation for small slopes that is made in the bibliography, with those corresponding to the exact case (without any approximation), as well as with the experimental results obtained in the laboratory using a steel ruler as a thin bar [6].

2 Presentation of the problem

In this paper we will present the Euler - Bernoulli model for the transverse vibrations of a homogeneous and flexible bar. In the case we will study, the bar of length L is embedded in one end and free at the other. We will denote by $w(x, t)$ the distance, at time t, from the point x_2 [0, L] to the equilibrium position of the bar. When an upward force is applied at its free end, the bar is displaced by drawing a curve in relation to its equilibrium position. From this position profile, when leaving the rest, the bar will describe a vibration movement as shown in Fig. 1.

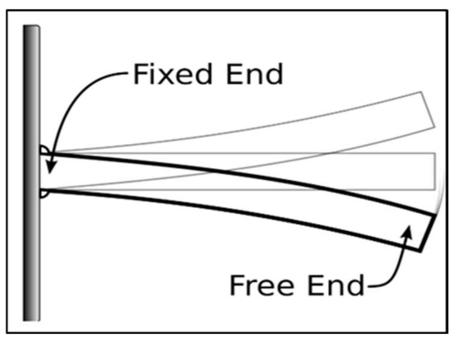

Figure 1: Bar with a force applied to its free and fixed end.

In the deduction of the Euler - Bernoulli equation, the following hypotheses are used:

- Consider straight elements (prisms) whose length is much greater than other measures.
- The materials are homogeneous linear elastics governed by Hooke's Law [7].
- The *xy* plane is a plane of symmetry of the bar and all loads act on this plane (flexion plane).
- The cross section of the bar has at least one axis of symmetry, coinciding with the vertical axis.

- The bar has a straight central axis coinciding with the *x* axis.
- We only consider deformations due to pure bending, so any sharp deformations were neglected.
- Sections perpendicular to the *x*-axis remain flat after deformation.
- At first, we will start by describing the equation of the deflection curve of this bar in a static way and then through the analysis of the equilibrium equations of an infinitesimal element we will arrive at the proposed equation.

3 Methodology

3.1 Development of the Rotor Mathematical Model by Means of the Euler-Bernoulli Theory

The equations of the Euler-Bernoulli beam are also known as the classical theory of beams [8], simple equations that allow obtaining the deformation in bars, both for bending, tension and torsion. It is necessary to make several guesses and/or approximations [9]. Using relationships between bending moment, the properties of the cross section, internal forces and deformations and implementing all these conditions in the equations of balance of forces and internal and external moments, a differential equation of several variables is obtained; in this case two variables, second order in time and fourth order in displacements. You need to make assumptions about deformation in order to make the change from a statically indeterminate problem to a statically determined problem. With what then proceeds to find the relationships between stress-strain and corroborate the equilibrium conditions [10]. A horizontal cross-section beam is considered for this work (Fig. 2.A); of constant cross section with an axis of symmetry; considering that when the beam is not subject to any load, that is, it is not deformed, a horizontal line joins all the centroids of the cross sections.

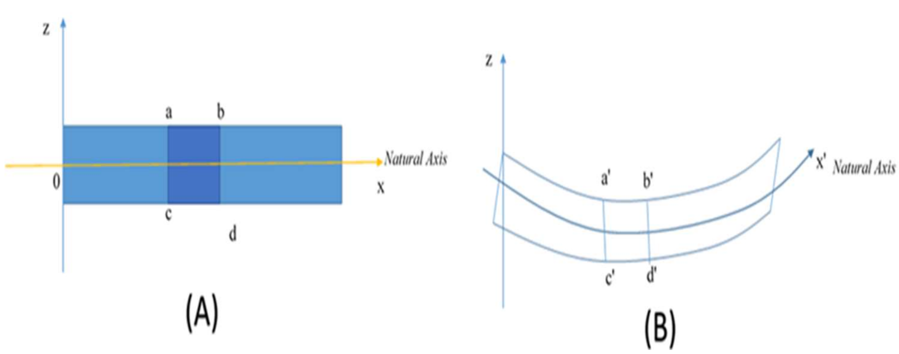

Figure 2: Beam with an applied load P and its deflection curve

In other words, the beam is cut into slices and each one of these slices will have its centroid in the same place joined by that drawn line, which is known as the line of the neutral axis of the beam. In (Fig. 2.A) a slice of the beam is considered in lateral view formed by two planes perpendicular to the neutral axis limited by the figure formed by *abcd*. When this beam element is subjected at each end to the application of a moment of equal magnitude but in the opposite direction, the original element flexes in the plane of symmetry, and the initially perpendicular planes are tilted a little (Fig. 2.B), causing the lines ad and *bc* become *a'd'* and *b'c'* but remain straight. This consideration is the principle of the theory of bending for beams, which in more adequate words is: the planar sections of the beam, normal to the neutral axis, remain planar despite the fact that the beam has been subjected to bending.

When the bar is flexed, longitudinal lines become curved and transverse lines rotate despite being straight. Thus, there is not only deflection along the axis but also a rotation. The angle of rotation θ is the angle between the axis and the tangent to the deflection curve. Analysing Fig. 3, it can be seen that the rotation angle in m_2 is $\theta + \theta d$. Thus, d is the infinitesimal angle between the normal plotted with respect to the tangents of the points.

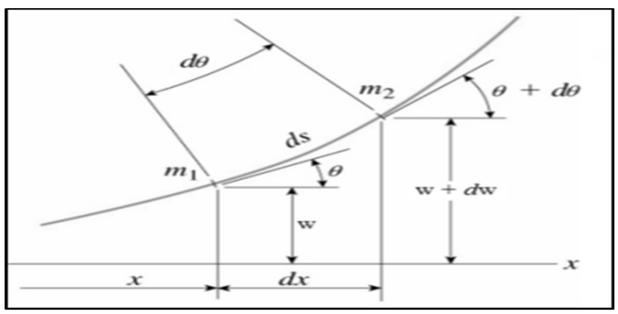

Figure 3: Bar deflection curve analysis.

For the triangle obtained in $O'\, m_1\, m_2$ in Fig. 4 we have to:

$\rho d\theta = ds$ (1)

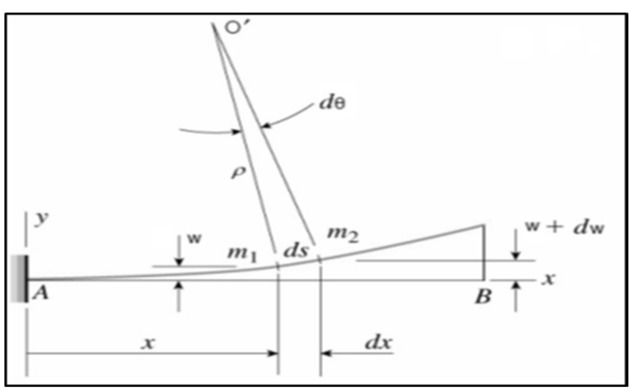

Figure 4: Deflection curve of a bar.

Where ds the infinitesimal is distance between m_1 & m_2 along the deflection curve and d is measured in radians. Another point to be analyzed in the bar is its curvature. If the force applied to the bar is too small, the bar will undergo minimal deformation and thus be practically straight. Its radius of curvature will be very large and the curvature will be very small. Otherwise, the flexion will be greater due to a greater force and then we will have a smaller radius of curvature and greater curvature. We then realize that the curvature is defined as the inverse of the radius of curvature and is given by the equation below:

$$k=1/\rho \qquad (2)$$

In this way, we can write equation Eq. 1 in terms of Eq. 2 and thus obtain

$$k=1/\rho=d\theta/ds \qquad (3)$$

As a convention of the signs for the bending moment, shear force and curvature (Fig. 5), we used the same one treated in Shihab et al (2013) [11].

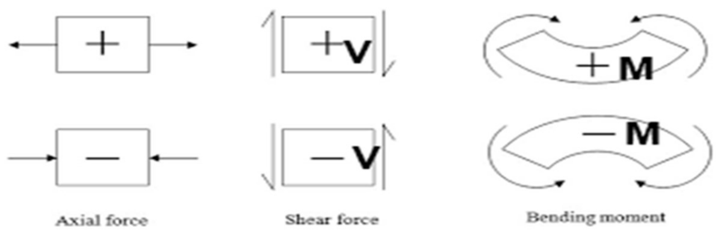

Figure 5: Signal Convention for M and V & Convention of signs for curvature.

3.2 Euler-Bernoulli equation solutions

Consider a thin bar under bending of length L and constant rectangular section of base b and height h, embedded at one end and subjected to a vertical force F at the free end, as seen in Fig. 6. It is assumed that the material of the bar is linear elastic, that its length is much greater than the lateral dimensions of the bar, that the length of the bar does not change (strictly speaking the length of its neutral line and that the deformations are small. Under these hypotheses we can use the Euler-Bernoulli relationship between the bending moment M of the applied force and the radius of curvature r of the deformed bar [12] see Eq. 4:

$$M=EI/p \qquad (4)$$

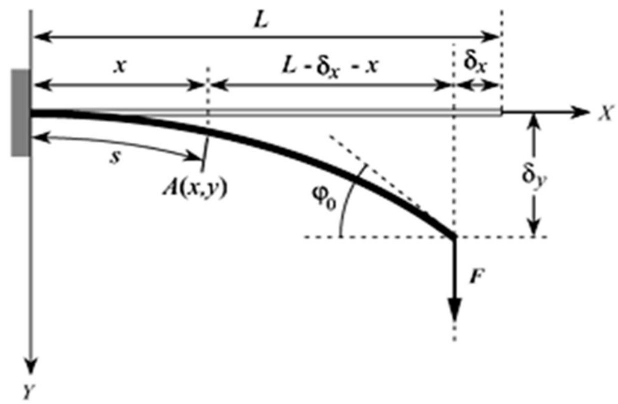

Figure 6: Thin bar recessed at one end and with concentrated vertical force at the free end. Definition of parameters.

Where *E* is the Young's modulus of the material, *I* is the moment of inertia of the cross section of the bar and r is the radius of curvature. The *EI* product, which depends on the type of material used and the geometric characteristics of the bar section, is called the "bending stiffness modulus" of the bar or simply "stiffness".

The radius of curvature *r* of a curve with equation *y = y(x)* can be calculated using Eq. 5:

$$\frac{1}{p} = \frac{d^2\, y/d\, x^2}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \qquad (5)$$

In the particular case in which the displacement of the points of the bent bar with respect to those of the unbent bar is small, that is, in the case of small slopes, it is possible to neglect the term $(dy/dx)^2$ against the unit in Eq. 5, so the radius of curvature can be calculated by the expression Eq. 6:

$$\frac{1}{p} = \frac{d^2 y}{d\, x^2} \qquad (6)$$

Which allows writing the differential equation of the elastic curve for a thin bar of linear elastic material under the hypothesis of small displacements (elastic of small slopes) in the form Eq. 7:

$$\frac{d^2 y}{d\, x^2} = M/EI \qquad (7)$$

Which is a second order linear differential equation. So far there is no difference to how the problem is treated in all the books of the university courses of Physics and Mechanics. The modification to be introduced in this work is related to the calculation of the bending moment *M* and is based on the fact that the bar is inextensible (its neutral fiber), which implies that the length of the bar does not change due to the flexion. To obtain the equation of the elastic curve

$y = y(x)$ for small displacements of the bar, it is necessary to integrate the differential Eq. 7 and for this it is necessary to know the bending moment M. Eq. 4.3 shows a photograph of a bar thin of length L embedded at one end on which a concentrated vertical force F is applied at the free end. Since the length of the deformed bar is also L, the free end of the bar undergoes both a vertical displacement δy and a horizontal displacement δx. Normally, and as it appears in Feynman's article and in other texts, the horizontal displacement δx is assumed to be zero, which is not compatible with the fact that the length of the bar does not change after deformation. The bending moment M due to the point load F applied at the end of the bar with respect to the section located at a distance x from the embedment is given by the equation Eq. 8 from (Fig. 6):

$$M(x)=F(L - \delta x - x) \qquad (8)$$

and substituting Eq. 8 in the differential Eq. 7 of the elastic for small displacements (small slopes), it takes the form Eq. 9:

$$\frac{d^2y}{d\,x^2} = \frac{F}{EI}(L - \delta x - x) \qquad (9)$$

Where for the cantilevered bar the boundary conditions are Eq. 10 and Eq.11:

$$y(0)=0 \qquad (10)$$

$$(dy/dx)_{x=0} = 0 \qquad (11)$$

With the help of Eq. 10 and Eq. 11 it is possible to easily integrate Eq. 12, obtaining:

$$\frac{dy}{dx} = \frac{F}{2EI}\left((2L - \delta x)x - x^2\right) \qquad (12)$$

$$y(x)=F/6EI[3(L - \delta x)x^2 - x^3) \qquad (13)$$

Eq. 13 allows calculating the maximum vertical displacement of the bar, δy, which occurs at the free end of the bar. Its value is obtained, as can be deduced from Fig.7, substituting $x = L - \delta x$ in Eq. 13:

$$\delta y=F/3EI(L - \delta x)^3 \qquad (14)$$

Eq. 12 provides the value of the slope of the elastic for each value of the x coordinate, while Eq. 13 is the Cartesian equation $y = y\,(x)$ of the elastic of the bar. It is important to note that the Eq. 12 of the elastic is valid as long as the square of the slope of the elastic is much less than unity, since that has been the hypothesis to obtain Eq. 7, that is, it must the condition is met:

$$\left(\frac{dy}{dx}\right)^2 \ll 1 \tag{15}$$

As the maximum slope occurs at the free end of the bar, that is, for the point of abscissa $x = L - \delta x$, taking into account Eq. 12 evaluated at that point, the validity condition of the equations for small pending is:

$$\left(\frac{F}{2EI}\right)^2 (L - \delta x) \ll 1 \tag{16}$$

To know whether or not the approximation of small slopes is applicable in a practical case, condition in Eq. 16 should be verified. However, sometimes it is possible that the value of the Young's modulus of the material, E, is unknown, since what is intended with the experimental study may be its determination. In this case, it is easy to write Eq. 16 without showing the value of EI. To do this, EI is cleared from Eq. 14 and replaced in Eq. 16, obtaining the applicability condition of the equations for small slopes as a function of the horizontal and vertical displacements of the free end, δx and δy, respectively which can be easily measured in the laboratory. From Eq. 16 we obtain Eq. 17:

$$\frac{9\delta^2 y}{4(L - \delta x)^2} \ll 1 \tag{17}$$

However, to evaluate Eq. 12 to Eq. 14 it is not only necessary to know Young's modulus, E, the moment of inertia, I, and the applied force, F, but also the horizontal displacement, dx, of the end free of the bar, and this is an unknown in the problem, since it is in turn a function of E, I and F. It is here that the hypothesis is used that the bar is inextensible and, therefore, its length L does not change (strictly speaking, the length of its neutral fiber, although if the bar is thin it can be said that both coincide). To introduce this new hypothesis, it is enough to take into account the relationship that exists between the arc element ds and its Cartesian components δx and δy as Eq. 18:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{18}$$

so the length L of the bar will be:

$$L = \int_0^{L - \delta x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{19}$$

where the upper limit of integration takes into account the fact that the maximum value of the x coordinate of the elastic bar is not L, as indicated in the bibliography for the case of approximation of small slopes, but $L - dx$, as seen in Fig.7. Substituting Eq. 12 in Eq. 19, we obtain Eq. 20:

$$L = \int_0^{L-\delta x} \sqrt{1 + \left(\frac{F}{2EI}\right)^2 [2(L - \delta x)x - x^2]^2 dx} \qquad (20)$$

Eq. 20 allows calculating the value of the horizontal displacement of the free end of the bar when the values of *L*, *E*, *I* and *F* are known. Once the value of *dx* is obtained, it is possible to determine the rest of the parameters that characterize the bent bar, in particular δy and the Cartesian equation of elastic. If $dx = 0$ is taken in the previous equations, the following values of $y(x)$ and δy are obtained Eq. 21:

$$y(x)= F/6EI(3Lx^2 - x^3) \qquad (21)$$

$$\delta y = FL^3/3EI \qquad (22)$$

Which are the ones that appear in the bibliography [13].

3.3 Numerical results

To compare the results obtained with the approximation for small slopes that has been presented in this work with the corresponding approximation that is made in the bibliography (Eq. 21 and Eq. 22), as well as with the case in which it is solved the differential Eq. 4 without considering any approximation in the radius of curvature [14], it is convenient to define a series of dimensionless parameters. To do this, the dimensionless force parameter is introduced:

$$k=FL^2/EI \qquad (23)$$

the dimensionless coordinates *u* and *v*:

$$u = \frac{x}{L}; \quad v = \frac{y}{L} \qquad (24)$$

As well as the horizontal and vertical dimensionless displacements of the free end of the bar, *uf* and *vf*, respectively, defined by the equations:

$$uf = \frac{\delta x}{L}; \quad vf = \frac{\delta y}{L} \qquad (25)$$

Using Eq. 23 to Eq. 25, Eq. 13, Eq. 14 and Eq. 20 can be written in the form:

$$v = \frac{k}{6}[3(1 - uf)u^2 - u^3] \qquad (26)$$

$$vf = k/3(1 - uf)^3 \qquad (27)$$

$$\int_0^{1-uf} \sqrt{1+\frac{k^2}{4}[2(1-uf)u-u^2]^2}du = 1 \quad (28)$$

Likewise, Eq. 16 of validity of the approximation for small slopes will now be written:

$$\frac{k^2}{4}(1-uf)^4 \ll 1 \quad (29)$$

Eq. 28 allows to obtain uf as a function of the dimensionless parameter *k*. However, the unknown *uf* appears both in the integrand and in one of the limits of integration. To obtain the value of *uf*, it is proposed to follow a very interesting and instructive trial-error procedure for the students. First, a function *g* of the variable *uf* is defined as follows:

$$g(uf) = \int_0^{1-uf} \sqrt{1+\frac{k^2}{4}[2(1-uf)u-u^2]^2}du - 1 \quad (30)$$

With the help of this function, Eq. 28 can be written as Eq. 31:

$$g(uf) = 0 \quad (31)$$

At this time, use is made of the fact that university students are usually familiar with software such as Mathematical, of which there is also a version for students. For this reason, the integral of Eq. 30, for each value of *k*, can be solved using the numerical integration command "N Integrate" of the Mathematical program. Fig. 7 shows the results obtained for the case *k = 0.5*. As can be seen, there is a solution of Eq. 31 that is determined by a trial-error procedure by varying the value of uf until *g(uf)* is less than 10^{-8}, which is taken as *0* for computational purposes.

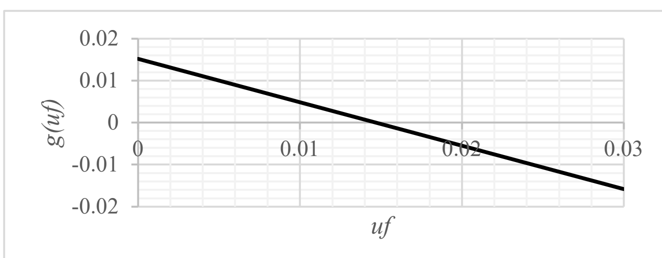

Figure 7: Function *g(uf)* for *k*

Fig. 8 shows the results obtained from uf as a function of *k* compared to the exact ones calculated without making any approximation, solving the elliptic integrals that appear in this case. It is important to note that in the approximation made in all university books that address the problem of bending a bar for the case of small slopes, *uf = 0* is taken as the starting hypothesis, since it is assumed that the force *F* is always applied at the point of abscissa *x = L*. In Fig. 9 the relative errors that are committed when using the approximation that have been considered in this work and that made in the bibliography, with respect to the exact values obtained when no approximation is made. For the approximation considered in this work, the relative error is below 10% for $k < 0.8$, while with the approximation that appears in all books the relative error is 100%.

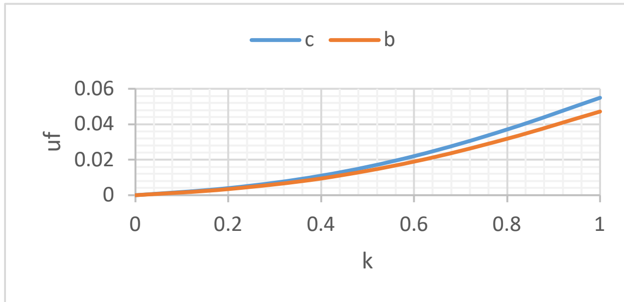

Figure 8: Values of the dimensionless horizontal displacements of the free end of the bar as a function of *k*: Approximation for small slopes carried out in the bibliography a (a), approximation carried out in this work (b) and exact values

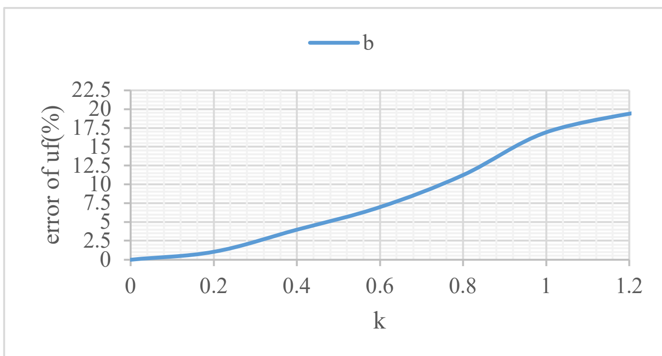

Figure 9: Relative errors of the dimensionless horizontal displacements of the free end of the bar as a function of *k*: Approximation for small slopes carried out in the bibliography (a) and approximation carried out in this work

Once the value of *uf* has been determined for each *k*, it is possible to calculate the dimensionless vertical displacement of the free end of the bar, *vf*, simply by using Eq. 26. Fig. 10 shows the results obtained compared with the exact ones and with the approximate ones for small slopes obtained using Eq. 22 that is shown in all the books and that will now be written in the form $vf = k/3$. As can be seen from Fig. 10 the approximation for small slopes that has been presented in this work provides results more in line with the reality represented by the exact solution for a greater range of *k* values, since the relative error is below 10% for the *k* considered, while with the approximation that appears in the books this error is much greater. In Fig. 11 the square of the first derivative of the function *v* evaluated at the free end of the bar has been represented, that is, at the point of abscissa $u = 1 - uf$.

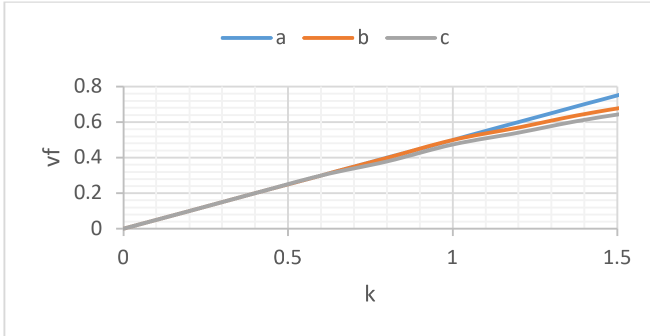

Figure 10: Values of the dimensionless vertical displacements of the free end of the bar as a function of k: Approximation for small slopes carried out in the bibliography a(a), approximation carried out in this work(b) and exact values(*c*).

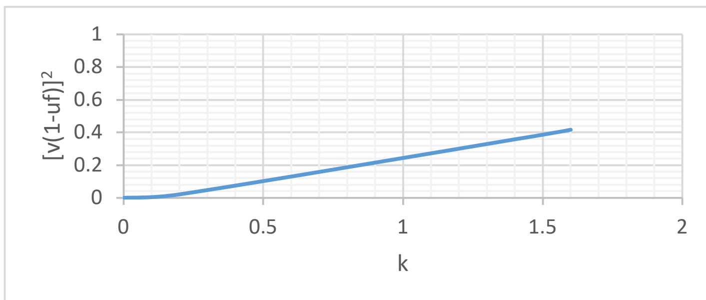

Figure 11: Square of the first derivative of the function v evaluated at the free end of the bar, that is, at the point of abscissa $u = 1 - uf$.

Finally, in Fig.12 the elastic obtained using Eq. 26 for k = 0.2, 0.4, 0.6 and 0.8 have been represented. As can be seen, both *uf* and *vf* are different from *0*.

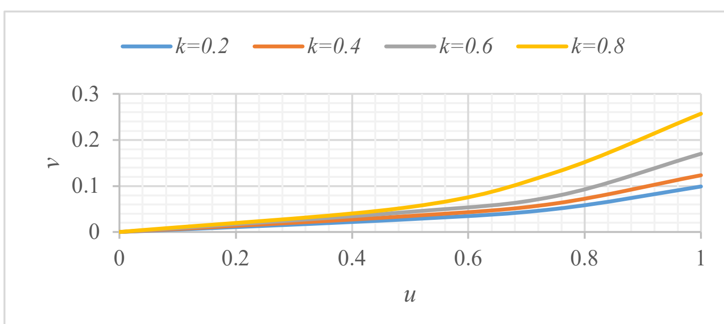

Figure 12: The dimensionless elastic for different values of *k* obtained with the approximation for small slopes presented in this work.

4 Experimental results and discussion

Fig. 13 shows a schematic image of the bar embedded at one end and on which a concentrated vertical force is applied at the free end. The bar is a steel rule of length L whose cross section is rectangular with base b and height h. The rule has been embedded with the help of a double nut of the kind found in all Physics laboratories, which has been attached to a vertical rod on a support. With the help of two metal rectangular sheets placed one above the ruler and the other below, the ruler is adjusted to the double nut. The values of b and h of the rectangular section of the thin bar have been determined with the help of a caliper and a palmer, respectively. Measurements of b and h have been taken every 5 *cm* along the rule and the mean of each of the series of measurements has been calculated. The absolute error of b and h has been taken as the greatest value of the sensitivity of the measuring device and the mean deviation of the measurements made, in both cases the sensitivity of the measuring instruments used being greater. We calculate the moment of inertia of the rectangular section using Eq. 4:

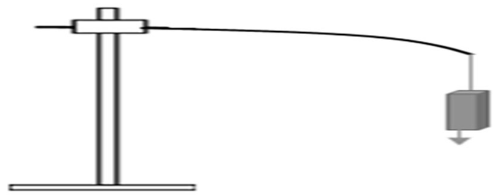

Figure 13: Schematic representation of the steel rule embedded in extreme experimentally analyzed.

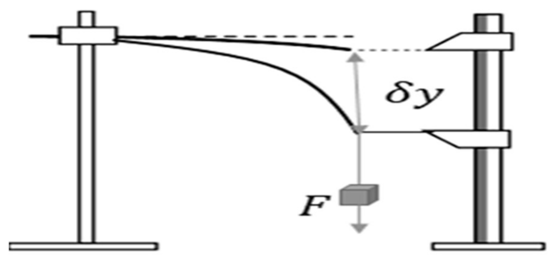

Figure 14: Experimental determination of the vertical displacement of the free end of the extruded bar due to the external force F applied.

With an electronic balance, the mass of the ruler is determined and taking $g = 9.80 \pm 0.01$ m/s^2 its total weight is calculated and, from this, the weight P for a length of the ruler of 30 cm. The maximum vertical displacement of the free end of the bar when subjected to different concentrated forces F was then measured, taking as a reference value the position of the free end when no force F is applied (Fig. 14). As the situation of small slopes is going to be analyzed, the differential equation to be considered is linear, which implies that the total vertical displacement of the free end of the bar will be the sum of the displacements due to the applied force and the own weight. When the maximum vertical displacement due only to the self-weight of the bar is very small (in the laboratory it has been experimentally measured that this vertical displacement is less than 8 *mm*) and also the slopes when applying external forces are small, it can be assumed, without make a significant mistake, that the displacement of the free end of the bar due to its own weight is independent of the applied force. In Fig. 15 the experimental values of the vertical displacements *dy* measured with respect to the position of the free end in the absence of applied force have been represented, as a function of the force *F*. As can be seen for small applied forces, the relationship between *dy* and *F* It is linear, but as the value of the applied force increases, the relationship between *dy* and *F* is no longer linear. Analyzing the zone of linear behavior, which can be taken for forces applied between 0 and 1.2 *N* (Fig. 16), the value of the stiffness of the bar has been determined by means of a least squares adjustment of Eq. 22, obtaining :

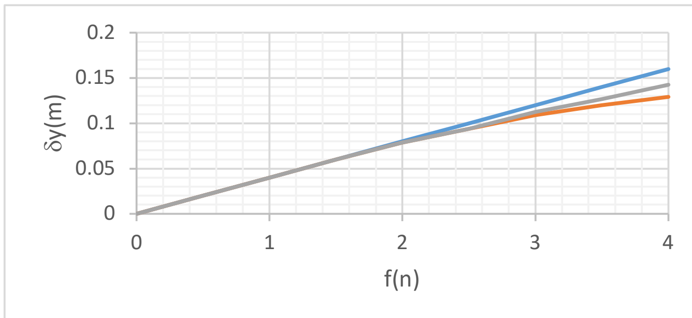

Figure 15: Experimental values of the vertical displacements of the free end of the bar as a function of the applied force *F*: Approach for small slopes carried out in the bibliography (straight line) and approximation presented in this job (curved line).

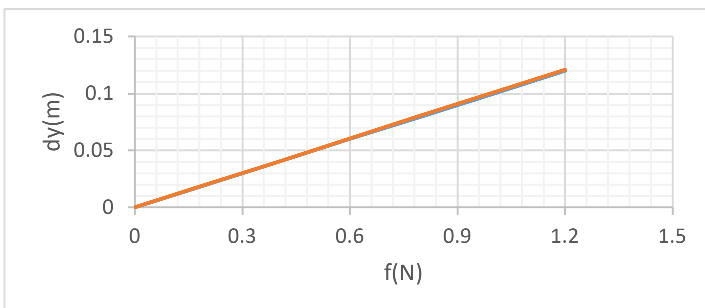

Figure 16: Experimental values of the vertical displacements of the free end of the bar as a function of *F* for the linear zone and the straight line of Eq. 22.

Which implies that the Young's modulus of the steel from which the rule is made is $E = 198 \pm 15$ GPa. This value agrees with the Young's modulus of steel that can be found in the literature and is around 200 *GPa* [15]. In Fig. 16 the values calculated theoretically using the approximations for small slopes of the literature Eq. 22 and the one considered in this work Eq. 14.

5 Conclusion

Starting from the hypothesis that the material from which a bent thin bar is made is linear elastic, the differential equation of the elastic of the bar has been presented in a manner similar to how it is done. This differential equation has been simplified considering the case of small displacements of the bar guideline (small slopes), but without making an approximation as extreme as is done in the university texts of Physics and Mechanics. An important aspect and of great educational interest is the way in which the condition that the bar is inextensible (the length of its neutral fiber does not change) has been applied, an aspect that is not considered in the aforementioned books. The equations of the elastic and the vertical and horizontal displacements of the free end of the bar have been obtained, comparing them with those corresponding to the general case of small and strong slopes, as well as with the approximation of small slopes that appears in the bibliography. In the latter case it is assumed that the horizontal displacement of the free end of the bar is zero. The theoretical results have been compared with the experimental ones obtained in the laboratory with a steel ruler, proving how the approximation for small slopes considered in this work is "less approximate" than the one that appears in the literature.

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