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# PERTURBATIONS OF WEYL'S THEOREMS FOR UNBOUNDED HYPONORMAL OPERATOR

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**Abstract.** The set of all hyponormal operators with a non-empty resolvent $\mathcal{R}\,(\mathcal{H})$ is suggested to develop different spectral properties such as (w), (gw), (b), and (gb). In this effort, we proceed to explore more properties in in the infinitely complex Hilbert space. We shall show that these operators satisfy various types of Weyl's and Browder's theorems. This work, dedicated to the assumptions of this class of operators, can give us new spectral properties, namely (am) and (gam), and then prove the a-Weyl theorem and the generalized a-Weyl's theorem. It also establishes a connection between the obtained property and Weyl's theorem for this operator. As application, we consider the space of analytic functions in a complex domain in the infinitely complex Hilbert space. We shall provide the sufficient conditions for these spaces to have the new properties (am) and (gam). A set of consequences is illustrated in the sequel.

**Keywords**: Unbounded Hyponormal operator, Weyl's type theorems, Browder's theorem, spectral properties.

## 1. Introduction

In Hilbert spaces, the theory of hyponormal operators has advanced significantly. For a clear explanation of the idea, see [1]. Researchers have attempted to apply the concept of hyponormality to unbounded operators in various cases. It turns out that limited and unbounded hyponormal operators have several characteristics. It is uncertain, nevertheless, if additional bounded hyponormal operator characteristics apply in the unbounded situation. In more recent work, the research is taken into account in many spaces for unbounded linear operators, including $L^2$ (see [2]), $L^p$ ( see [3]), Pick space (see [4]) and other investigation in [5].

In this effort, we keep looking at further Hilbert space properties. We'll show that these operators are consistent with various Weyl's and Browder's theorems. This study, which is focused on the assumptions made by this group of operators, might provide us two new spectral qualities, (am) and (gam), before demonstrating the a-Weyl theorem and the modified a-theorem. Weyl's It also creates a connection between the found property and Weyl's operator theorem. As an example, we take into account the Hilbert space's infinite complexity and the space of analytic functions in a complex domain.

The remainder of the essay is devoted to the following: The techniques, ideas, and theory that we will apply in our investigation are described in Section 2. The results and the originality are presented in Section 3. The implementation of this effort's idea is shown in Section 4.

## 2. Methodology

All through this work, $\mathcal{H}$ denotes to infinite dimensional complex Hilbert space, $C(\mathcal{H})$ is the set of all closed linear operators defined on $\mathcal{H}$. For an operator $A \in C(\mathcal{H})$, we define $N(A)$ as the kernel of $A$, while $\Delta(A)$ represents the domain, and $R(A)$ denotes the range of A. The upper semi Fredholm operator is define if $R(A)$ is closed and $\alpha(A) = \dim N(A)$ is finite while we say that $A$ is lower semi Fredholm operator if $B(A) = codim\ R(A)$ is finite. A Fredholm operator is upper and lower semi Fredholm operator.

$SF_+(\mathcal{H}) = \{A \in C(\mathcal{H}): A\ is\ upper\ semi\ Fredholm\}$ and

$SF_-(\mathcal{H}) = \{A \in C(\mathcal{H}): A\ is\ lower\ semi\ Fredholm\}$.

The index of $A$ is defined as ind $(A) = \alpha(A) - B(A)$.

An operator $A \in C(\mathcal{H})$ which is Fredholm operator of index 0 is defined as Weyl operator, while $\sigma_w\ (A) = \{\eta \in \mathbb{C} : A - \eta I$ is not weyl$\}$ is used to define the Weyl spectrum of $A$. In addition we can assign the following notations:

$SF_+^-(\mathcal{H}) = \{A \in C(\mathcal{H}): A \in SF_+(\mathcal{H}), ind(A) \leq 0 \}$

$SF_-^+(\mathcal{H}) = \{A \in C(\mathcal{H}): A \in SF_-(\mathcal{H}), ind\ (A) \geq 0\}$

In [6], Berkani generalized the concept of Fredholm operators to B-Fredholm operators as follows

$$\Omega(A) = \{i \in \mathbb{N}: \forall j \in \mathbb{N}, j \geq i \Rightarrow R(A^i) \cap N(A) \subseteq R(A^j) \cap N(A)\}$$

The degree of stable iteration of $A$ is denoted by $dis(A)$ and defined by $dis(A) = inf\ \Omega(A)$ and $dis(A) = \infty$ when $\Omega(A) = \varnothing$.

Furthermore, for $A \in C(\mathcal{H})$ the $B - Fredholm$ operator is *upper* and *lower* semi $B - Fredholm$ operator, where $A$ is upper (resp., lower) semi $B - Fredholm$ operator if $\exists\ d \in \Delta(A)$: $\dim\{N(A) \cap R(A^d)\}$ is finite and $R(A^d)$ closed and (resp., $codim\ \{R(A) + N(A^n)\}$ is finite), and the index of $A$ is

$$\text{ind}(A) = dim\{N(A) \cap R(A^d)\} - codim\{R(A) + N(A^d)\}.$$

We call $A \in C(\mathcal{H})$ as $B - W$eyl if it's $B$- Fredholm operator with $index$ 0 and $\sigma_{\text{BW}}(A)$ is used to symbolize the $B$-$W$eyl spectrum of $A$ and defined by $\sigma_{\text{BW}}\ (A) = \{\eta \in \text{C} : A - \eta$ I is not $B$-weyl $\}$.

Moreover, the ascent $\mathcal{M}(A)$ and descent $\aleph(A)$ for $A \in C(\mathcal{H})$ are defined as:

$$\mathcal{M}(A) = \inf\{d: N(A^d) = N(A^{d+1})\}$$
$$\aleph(A) = \inf\{d: R(A^d) = R(A^{d+1})\}$$

An operator $A \in C(\mathcal{H})$ is called Browder if it's both upper and lower semi Browder, where $A \in C(\mathcal{H})$ is upper semi- Browder if $\mathcal{M}(A) < \infty$ with $A$ is upper semi- Fredholm and it is lower semi-Browder if $\aleph(A) < \infty$ with $A$ is lower semi – Fredholm.

Now, we can define the following spectrum for an operator $A$ as:

$\sigma_{\text{ub}}(A) = \{\eta \in \mathbb{C}: A - \eta$I not upper semi-Browder $\}$,

$\sigma_{\mathrm{lb}}(A) = \{\eta \in \mathbb{C}: A - \eta\mathrm{I}$ not lower Semi-Browder$\}$ and

$\sigma_b(A) = \{\eta \in \mathbb{C}: A - \eta\mathrm{I}$ not Browder $\}$, respectively.

Weyl claims that the Wey'l spectrum of a hermition operator contains exactly all of the points in the spectrum of $A$ with the exception of those points, which are isolate eigenvalues of restricted pluralism, in [7], where he proved the Weyl's theorems for bounded hermition operators. Weyl's theorem has now been extended to other types of bounded operators [8]. The generalization of conventional Weyl's theorem has given by Berkani in [9] when he show that if A is normal operator belong to $L(\mathcal{H})$ then $\sigma_{\mathrm{BW}}(A) = \sigma(A) \setminus E(A)$, where $E(A)$ is the set of all isolated eigenoalues of $A$. Also, Berkani proved this generalized for hyponormal operator which belong to $L(\mathcal{H})$.

Newly, this research has extended to the unbounded hyponormal operators [10] and the unbounded posinormal operator [11].

Throughout the paper, we define $\mathcal{R}(\mathcal{H}) = \{A \in C(\mathcal{H})$: $A$ is unbounded hyponormal operator with $\rho(A) \neq \varnothing$. In [11] many Weyl's type theorems, properties and the spectrum for $A \in \mathcal{R}(\mathcal{H})$ are established, so that an operator $A \in \mathcal{R}(\mathcal{H})$ are Polaroid with ascent less than or equal to 1.

In this paper, we present the (am) and (gam) properties for $A \in C(\mathcal{H})$ and prove that if $A \in \mathcal{R}(\mathcal{H})$ then $A$ possess $(am)$ and (gam) properties. Furthermore, we give the relation between them. Next section studies the a-Weyl's theorem and generalized a-Weyl's theorem for $A \in R(\mathcal{H})$ and then define the relation between Weyl's type theorems. Lastly, we summarize the results in a diagram.

### 2.1. Spectrum of A $\in$ R$(\mathcal{H})$

Definition 2.1. ([11,12]).

An operator $A \in C(\mathcal{H})$ is called hyponormal if (1) $\Delta(A) = \Delta(A^*)$ (2) $A$ is closed and (3) $A^*A - AA^* \geq 0$.

Also, we can define the unbounded hyponormal operator if the following conditions are hold

$A^*$ exists, $\Delta(A) = \Delta(A^*)$, $\Delta(A)$ is dense in $\mathcal{H}$ and

(ii) $\| Ax \| \geq \| A^*x \|, \forall x \in \Delta(A) = \Delta(A^*)$.

The following theorem is not true for bounded hyponormal operator.

Theorem 2.2. ([11])

Let $A \in \mathcal{R}(\mathcal{H})$, then $\mathcal{M}(A - \eta I) = 0$ or $1$ for all $\eta \in \mathbb{C}$.

Theorem 2.3. ([13]).

Let $A \in C(\mathcal{H})$

1. A has SVEP at $\eta$, If $\mathcal{M}(A - \eta I) < \infty$ for some $\eta \in \mathbb{C}$.
2. If $A$ is surjective and not injective, then $A$ not possess SVEP at $\eta = 0$.

Definition 2.4 ([14]).

For $A \in C(\mathcal{H})$, $\eta \in iso\ \sigma(A)$ is called pole of order $m$ if $m = \mathcal{M}(A - \eta I) < \infty$ and $\aleph(A - \lambda I) < \infty$.

Lemma 2.5. ([11]).

If $A \in \mathcal{R}(\mathcal{H})$, then $\eta$ is a simple pole of the resolvent of A iff $\eta$ is an isolated point of $\sigma(A)$.

Remark 2.6 ([15])

We will use $\sigma_a(A)$ to define the approximate point spectrum of $A$ and $\sigma_{\mathrm{LD}}(A)$ to the Left Drazin invertible spectrum of $A$, $\sigma_D(A)$ be the Drazin spectrum of $A$.

Definition 2.7. ([11])

An operator $A \in C(\mathcal{H})$, satisfies:

(i) Weyl's theorem if $\sigma(A) \setminus \sigma_w(A) = E_0(A)$.

(ii) Generalized Weyl's theorem if $\sigma(A) \setminus \sigma_{\mathrm{Bw}}(A) = E(A)$.

(iii) Browder's theorem if $\sigma_w(A) = \sigma_b(A)$.

(iv) Generalized Browder's theorem if $\sigma(A) \setminus \sigma_{\mathrm{Bw}}(A) = \pi(A)$.

(V) a-Browder's theorem if $\sigma_{SF_+^-}(A) = \sigma_{\mathrm{ub}}(A)$.

(VI) a-Weyl's theorem if $\sigma_a(A) \setminus \sigma_{SF_+^-}(A) = E_a^o(A)$.

(VII) Generalized a-Weyl's if $\sigma_a(A) \setminus \sigma_{\mathrm{SB}F_+^-}(A) = E_a(A)$.

(VIII) Property (w) if $\sigma_a(A) \setminus \sigma_{SF\mp}(A) = E_0(A)$.

(Ix) Property $(gw)$ if $\sigma_a(A) \setminus \sigma_{\mathrm{SB}F_+^-}(A) = E(A)$.

(x) Property $(b)$ if $\sigma_a(A) \setminus \sigma_{SF\mp}(A) = \sigma(A) \setminus \sigma_b(A)$.

(XI) Property $(gb)$ if $\sigma_a(A) \setminus \sigma_{\mathrm{SB}\ _+^-}(A) = \pi(A)$.

Definition 2.8. ([16])

We say that $A \in L(\mathcal{H})$, is having property (am) if $\sigma_a(A) \setminus \sigma_b(A) = E_a^0(A)$.
Definition 2.9. ([16])
For $A \in L(\mathcal{H})$, A is possessing property (gam) if $\sigma_a(A) \setminus \sigma_D(A) = E_a(A)$.
Theorem 2.10. ([17])
If both $\mathcal{M}(A)$ and$\aleph(A)$ are finite then $\mathcal{M}(A) = \aleph(A)$.
Theorem 2.11. ([17])
If A is a linear operator defining on a vector space $X$ then:

1. $\alpha(A) \leq \beta(A)$, if $\mathcal{M}(A) < \infty$
2. $\beta(A) \leq \alpha(A)$ If $\aleph(A) < \infty$;
3. If $\mathcal{M}(A) = \aleph(A) < \infty$ then $\alpha(A) = \beta(A)$ (possibly infinite);
4. If $\alpha(A) = \beta(A) < \infty$ and if $\mathcal{M}(A)$ or $\aleph(A)$ is finite then $\mathcal{M}(A) = \aleph(A)$.

Corollary 2.12. ([17])

Let $A \in L(X)$ where $X$ is Banach space. We have

1. If $A$ has the SVEP then $\sigma_{\text{SU}}(A) = \sigma(A)$ and $\sigma_{\text{Se}}$ (A) = $\sigma_{\text{ap}}$ (A).
2. If $A^*$ has the SVEP then $\sigma_{\text{ap}}(A) = \sigma(A)$ and $\sigma_{\text{Se}}(A) = \sigma_{\text{SU}}(A)$.
3. If both $A$ and $A^*$ have the SVEP then $\sigma(A) = \sigma_{\text{SU}}(A) = \sigma_{\text{ap}}(A) = \sigma_{\text{Se}}(A)$.

## 3. Results in $\mathcal{R}(\mathcal{H})$

This section indicates the main outcomes of this effort. We shall start with the following suggested property:

Definition 3.1.

An operator $A \in C(\mathcal{H})$ is said to have property $(am)$ if $\sigma_a(A) \setminus \sigma_b(A) = E_a^0(A)$.

Theorem 3.2.

If $A \in \mathcal{R}(\mathcal{H})$, then A satisfies (am) property.

Proof: Let $\lambda \in \sigma_a(A) \setminus \sigma_b(A)$. Then $(A - \eta I)$is Browder's operator. By Theorem 2.10 we have $\mathcal{M}(A - \eta I) = \aleph(A - \eta I)$, by theorem 2.11 (i), (ii) we have $\alpha(A - \eta I) = \beta(A - \eta I) < \infty$.

Now, if $\alpha(A - \eta I) = \beta(A - \eta I) = 0$ the $(A - \eta I)^{-1}$ closed and bounded, thus $\eta \notin \sigma(A)$, which is contradiction then $\alpha(A - \eta I) > 0$.

Since $A \in \mathcal{R}(\mathcal{H})$ is polaroid operator and $1 = \mathcal{M}(A - \eta I) = \aleph(A - \eta I)$ then $\eta$ is isolated point of $\sigma(A)$, thus $\eta \in E_0^a(A)$.

Now, let $\eta \in E_0^a(A)$, $A \in \mathcal{R}(\mathcal{H})$, then, $0 < \mathcal{M}(A - \eta I) = \aleph(A - \eta I) < \infty$, by Theorem 2.11 (i), (ii), we have $\alpha(A - \eta I) = \beta(A - \eta I) < \infty$, thus, $(A - \eta I)$ is Browder's operator i.e., $\eta \notin \sigma_b(A)$.Then, A is satisfies (am) property.

Another interesting property can be seen in the next definition.

Definition 3.3.

An operator $A \in C(\mathcal{H})$ is said to have property (gam) if $\sigma_a(A) \setminus \sigma_D(A) = E_a(A)$.

Theorem 3.4.

Let $A \in \mathcal{R}(\mathcal{H})$ and $A^*$ has SVEP, then, $A$ obey (gam) property.

Proof: Let $\lambda \in \sigma_a(A) \setminus \sigma_D(A)$ Since $A^*$ has SVEP. then by Theorem 2.10 (iii) we have $\sigma(A) = \sigma_{\text{ap}}(A)$, thus, $\lambda \in E_a(A)$.

Now, let $\lambda \in E_a(A)$ then $\lambda \in \sigma_a(A)$. But $A \in \mathcal{R}(\mathcal{H})$ then $(A - \lambda T)$ has finite ascent and descent so that $\lambda \notin \sigma_D(A)$, then we have $\lambda \in \sigma_a(A) \setminus \sigma_D(A)$ then, A obeys (gam) property.

Theorem 3.5.

Suppose $A \in \mathcal{R}(\mathcal{H})$, if A has property (gam) then, $A$ has property (am).

Proof: this proof can be done similarly to the proof of bounded case.

## 4. Results in the Weyl's theorem

We proceed to check the sufficient conditions on the operator $A \in \mathcal{R}(\mathcal{H})$ to satisfy the Weyl's theorem and it is various. We have the following result

Theorem 4.1.

If $A \in \mathcal{R}(\mathcal{H})$, then, A satisfies a-Weyl's theorem.

Proof: as $A \in \mathcal{R}(\mathcal{H})$, then by theorem 3.2. $A$ owns (am) property. Let $\eta \in \sigma_a(A) \setminus \sigma_{SF_+^-}(A)$
Since $\sigma_a(A) \setminus \sigma_b(A) = E_0^a(A)$ and $\sigma_{SF_+^-}(A) \subseteq \sigma_b(A)$, then $\eta \in E_0^a(A)$.

Now, Let $\eta \in E_0^a(A)$ then by definition of $E_0^a(A)$ , $\eta \in$ iso $\sigma_a(A)$, so that $\eta$ is a pole of the resolvent of $A$, then, $0 < \mathcal{M}(A - \eta I) = \aleph(A - \eta I) < \infty$ then, $\eta \in \pi_a(A) = \sigma_a(A) \setminus \sigma_{\mathrm{LD}}(A)$. Thus, $A$ satisfies a-Weyl's theorem.

Theorem 4.2.

Let $A \in C(\mathcal{H})$ and $\eta \in iso\ \sigma_a(A)$. Then $\eta \in \pi^a(A)$ if and only if $\eta \notin \sigma_{\mathrm{SB}\ _+^-}(A)$ . In addition $\eta \in \pi_0^a(A)$ if and only if $\eta \notin \sigma_{SF_+^-}(A)$.

Proof: The proof of this theorem is similar to the bounded case.

Theorem 4.3.

Let $A \in \mathcal{R}(\mathcal{H})$. If A has the property (gam) then A satisfies the generalized a-Weyl's theorem.

Proof: Suppose $\eta \in \sigma_a(A) \setminus \sigma_{\mathrm{SB}F_+^-}(A)$ since A has (gam) property, then,$\eta \in \sigma_a(A) \setminus \sigma_D(A) = E_a(A)$, so that $\sigma_a(A) \setminus \sigma_{\mathrm{SB}F_+^-}(A) \subset E_a(A)$.

For the opposite inclusion, suppose $\eta \in E_a(A)$, since $A$ possess (gam) property then, $\eta \notin \sigma_D(A)$ $\ \lambda \in \pi(A) \subset \pi^{a}(A)\mspace{2mu}$ then by theorem 4.2. $\eta \notin \sigma_{\mathrm{SB}F_+}(A)$. Thus, $A$ satisfies generalized a-Weyl's.

Corollary 4.4.

If $A \in \mathcal{R}(\mathcal{H})$ achieves the generalized a-Weyl's then A satisfies a-Weyl's theorem.

Corollary 4.5.

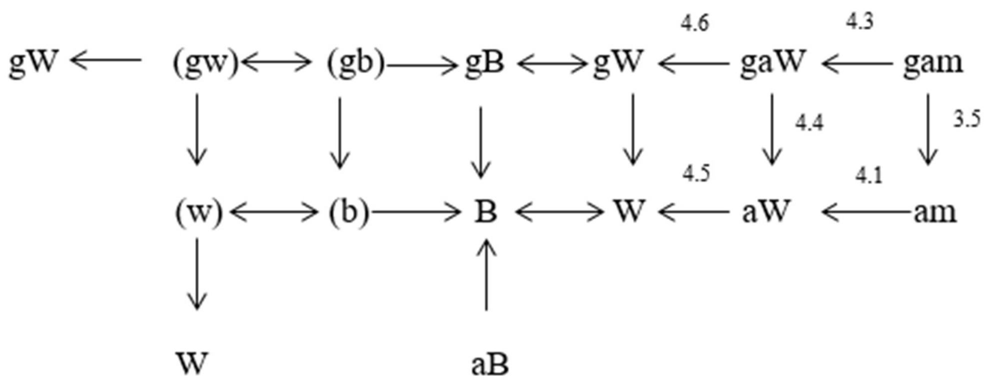
If $A \in \mathcal{R}(\mathcal{H})$ satisfies the a-Weyl's theorem then A satisfies Weyl's theorem.

## 6. Conclusion

The results in this paper can be summarized in a diagram, we used the following notations $W, gW,$ (gw), (w), gaW and aW to attend that an operator $A \in \mathcal{R}(\mathcal{H})$ satisfies $Weyl's$ theorem, the generalized$Weyl's\ theorem$, (gw) property, (w) property, generalized $a$-weyl's theorm and $a$-weyl's theorem, respectively. Similar notations gB, aB, $B$, $(b)$ and $(gb)$ are used for generalized Browder's theorem, a-Browder's theorem, Browder's theorem, property $(b)$ and property (gb).

The next schema summarizes the link among all of present theorem and properties. The numbers references to the result proved for an operator which belong to $\mathcal{R}(\mathcal{H})$.

Outlines the relationship between all of the current theorem's and properties.

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